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Symmetry Analysis of the Doebner–Goldin Equations

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The paper discusses the application of *MathLie* in connection with Lie group analysis. The examined example is the $(1+1)$-dimensional case of the Doebner–Goldin equation after Madelung transform. The related Lie-algeras are calculated. We present the generators, commutator tables and adjoint representations from the algebras. Furthermore we discuss the reduction of an example to ordinary differential equations and solve it explicitly.

1 Derivation of the Doebner–Goldin equations

During the investigation of Borel quantization for S^1 Dobrev, Doebner and Twarock [1] derived a nonlinear Schrödinger equation of the form (here with $m = 1$, $\hbar = 1$)

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + V(\vec{x},t)\psi + \frac{i}{2}KR_2(\psi)\psi + \sum_{j=1}^{5} D_j R_j(\psi)\psi. \tag{1}$$

This equation is called Doebner–Goldin equation. Here, the $R_j(\psi)$ with $j \in \{1, 2, \ldots, 5\}$ are real–valued functionals of the real–valued density $\varrho = \bar{\varrho} = \psi\bar{\psi}$ and the real-valued current $\vec{j} = \bar{\vec{j}} = \frac{i\hbar}{2m}\left(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi\right)$. They are given by

$$R_1(\psi) = \frac{\nabla\vec{j}}{\varrho}, \quad R_2(\psi) = \frac{\Delta\varrho}{\varrho}, \quad R_3(\psi) = \frac{\vec{j}^2}{\varrho^2}, \quad R_4(\psi) = \frac{\vec{j}\nabla\varrho}{\varrho^2}, \quad R_5(\psi) = \frac{(\nabla\varrho)^2}{\varrho^2}. \tag{2}$$

To apply Lie theory, we have to write equation (1) as a system of real functions. To do this we use the Madelung transformation [5]

$$\psi = \sqrt{\varrho(\vec{x},t)}\exp(iS(\vec{x},t)), \qquad \bar{\psi} = \sqrt{\varrho(\vec{x},t)}\exp(-iS(\vec{x},t)). \tag{3}$$

Considering the $(1+1)$-dimensional case with the functionals (2) and the Madelung transformation (3) we get the following system of equations:

$$\begin{aligned} &-8\varrho^2 V(x,t) + 4i\varrho\varrho_t - 8\varrho^2 S_t - \varrho_x^2 - 8D_5\varrho_x^2 \\ &\quad + 4i\varrho\varrho_x S_x - 8\varrho D_1\varrho_x S_x - 8\varrho D_4\varrho_x S_x - 4\varrho^2 S_x^2 - 8\varrho^2 D_3 S_x^2 \\ &\quad - 4iK\varrho\varrho_{xx} + 2\varrho\varrho_{xx} - 8D_2\varrho\varrho_{xx} + 4i\varrho^2 S_{xx} - 8D_1\varrho^2 S_{xx} = 0. \end{aligned} \tag{4}$$

After separating equation (4) into real and imaginary part a system of differential equations in S and ϱ follows:

$$\begin{aligned} &\varrho_t + \varrho_x S_x - K\varrho_{xx} + \varrho S_{xx} = 0, \\ &(1+8D_5)\varrho_x^2 + 2\varrho\left(4D_1\varrho_x S_x + 4D_4\varrho_x S_x + (4D_2-1)\varrho_{xx}\right) \\ &\quad + 4\varrho^2\left(2V(x,t) + 2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}\right) = 0. \end{aligned} \tag{5}$$

By permutating the six parameters $\{K, D_1, D_2, \dots, D_5\}$ we receive 63 different model equations (see Table 1 below) of a nonlinear Schrödinger type equation called Doebner–Goldin–Madelung equation.

2 Symmetry analysis of the $(1+1)$-dimensional Doebner–Goldin–Madelung equations

In order to find the symmetry group of equations (5) we apply the algorithms described in a lot of textbooks (e.g. [2, 3, 4, 6, 7]). We look for an algebra of vector fields of the form

$$v = \xi[1]\partial_x \cdot + \xi[2]\partial_t \cdot + \phi[1]\partial_\varrho \cdot + \phi[2]\partial_S \cdot,$$

where $\xi[1]$, $\xi[2]$ are functions of x and t and $\phi[1]$ and $\phi[2]$ depend on $\{x, t, \varrho, S\}$.

These coefficients are determined from the requirement that the second prolongation of v should annihilate the equation on the solution set of the equation. This was done using the *Mathematica* program *MathLie* [2].

The application of this theory to the system (5) leads to the following result:

Table 1. Permutation of parameters.

Nr.	Equations	Infinitesimals	Operators
1	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
2	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 +$ $2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $x\partial_S\cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot$
3	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
4	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 +$ $2\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
5	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + (1 + 8D_5)\varrho_x^2 -$ $2\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $x\partial_S\cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot$
6	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2\partial_t\cdot + x\partial_x\cdot,$ $x\partial_S\cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot$
7	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
8	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $\varrho_x^2 + 2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
9	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
10	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
11	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0,$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
12	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $\varrho_x^2 + 2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
13	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 +$ $2\varrho(4D_4 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
14	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + (1 + 8D_5)\varrho_x^2 +$ $2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $x\partial_S\cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot$

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16	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) + \varrho_x^2 +$ $\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
17	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $(1 + 8D_5)\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
18	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
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21	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $4\varrho^2(2S_t + S_x^2) +$ $(1 + 8D_5)\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x$, $\xi[2] = k_1 + 2t(k_3 + k_6 t)$, $\phi[1] = (k_7 - 2k_6 t)\varrho$, $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\partial_S\cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot$, $x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot$
22	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
23	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4(D_1 + D_4)S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
24	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
25	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
26	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
27	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $(1 + 8D_5)\varrho_x^2 + 2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$,	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
28	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
29	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
30	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
31	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
32	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
33	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $(1 + 8D_5)\varrho_x^2 + 2(4D_2 - 1)\varrho\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
34	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
35	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
36	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
37	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $4\varrho^2(2S_t + S_x^2) + (1 + 8D_5)\varrho_x^2$ $2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x$, $\xi[2] = k_1 + 2t(k_3 + k_6 t)$, $\phi[1] = (k_7 - 2k_6 t)\varrho$, $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\partial_S\cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot$, $x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot$
38	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0$, $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$
39	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0$, $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x$, $\xi[2] = k_2 + 2k_4 t$, $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot$, $2t\partial_t\cdot + x\partial_x\cdot$, $\varrho\partial_\varrho\cdot$

Nr.	Equations	Infinitesimals	Operators
40	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $(1 + 8D_5)\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
41	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2)+$ $2\varrho(4D_4S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
42	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
43	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $8D_5\varrho_x^2 + 2\varrho(4D_1S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
44	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4D_1S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
45	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
46	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
47	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4D_1S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
48	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $(1 + 8D_5)\varrho_x^2 + 2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
49	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
50	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4D_1S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
51	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
52	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $2\varrho(4D_4S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
53	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $2\varrho(4D_4S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
54	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2) + 2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
55	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2)+$ $2\varrho(4D_4S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
56	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $2\varrho(4D_4S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
57	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
58	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
59	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4D_1S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
60	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
61	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
62	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $2\varrho(4D_4S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
63	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1 + D_4)S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot +x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$

We derive from Table 1 the existence of two different algebras A_1, A_2:

$$A_1 = \{\partial_S\cdot, \partial_t\cdot, \partial_x\cdot, 2t\partial_t \cdot +x\partial_x\cdot, \varrho\partial_\varrho\cdot\}.$$

The dimension of the first algebra A_1 is 5.

The following equations are related to A_1: 1, 3, 4, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63.

The second algebra A_2 has dimension 7 and is represented by

$$A_2 = \{\partial_t\cdot, \partial_x\cdot, 2t\partial_t\cdot + x\partial_x\cdot, \partial_S\cdot, \partial_S\cdot + t\partial_x\cdot, x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot, \varrho\partial_\varrho\cdot\}.$$

The related equations from Table 1 are 2, 5, 6, 14, 15, 21, 37. To calculate the operation of the group we have to solve the following system of ordinary differential equations:

$$\frac{\partial \tilde{x}^n}{\partial\varepsilon} = \xi^n\left[\tilde{x}^i(\varepsilon), \tilde{u}^\beta(\varepsilon)\right], \qquad \frac{\partial \tilde{u}^\alpha}{\partial\varepsilon} = \phi^\alpha\left[\tilde{x}^i(\varepsilon), \tilde{u}^\beta(\varepsilon)\right], \qquad \tilde{x}^n = x^n, \quad \tilde{u}^\alpha = u^\alpha \quad \text{for} \quad \varepsilon = 0.$$

If we do this for the algebra A_1, we find:

$k_1 \neq 0,\ k_2 = k_3 = k_4 = 0$

$$\frac{\partial\tilde{t}}{\partial\varepsilon} = 0 \implies \tilde{t} = t,\quad \frac{\partial\tilde{x}}{\partial\varepsilon} = 0 \implies \tilde{x} = x,\quad \frac{\partial\tilde{\varrho}}{\partial\varepsilon} = 0 \implies \tilde{\varrho} = \varrho,\quad \frac{\partial\tilde{S}}{\partial\varepsilon} = k_1 \implies \tilde{S} = k_1\varepsilon + S;$$

$k_1 = 0,\ k_2 \neq 0,\ k_3 = k_4 = 0$

$$\frac{\partial\tilde{t}}{\partial\varepsilon} = 0 \implies \tilde{t} = t,\quad \frac{\partial\tilde{x}}{\partial\varepsilon} = k_2 \implies \tilde{x} = k_2\varepsilon + x,\quad \frac{\partial\tilde{\varrho}}{\partial\varepsilon} = 0 \implies \tilde{\varrho} = \varrho,\quad \frac{\partial\tilde{S}}{\partial\varepsilon} = 0 \implies \tilde{S} = S;$$

$k_1 = k_2 = 0,\ k_3 \neq 0,\ k_4 = 0$

$$\frac{\partial\tilde{t}}{\partial\varepsilon} = k_3 \implies \tilde{t} = k_3\varepsilon + t,\quad \frac{\partial\tilde{x}}{\partial\varepsilon} = 0 \implies \tilde{x} = x,\quad \frac{\partial\tilde{\varrho}}{\partial\varepsilon} = 0 \implies \tilde{\varrho} = \varrho,\quad \frac{\partial\tilde{S}}{\partial\varepsilon} = 0 \implies \tilde{S} = S;$$

$k_1 = k_2 = k_3 = 0,\ k_4 \neq 0$

$$\frac{\partial\tilde{t}}{\partial\varepsilon} = k_4\tilde{x} \implies \tilde{t} = \frac{\sqrt{2}t + x}{2\sqrt{2}}e^{\sqrt{2}k_4\varepsilon} - \frac{x - \sqrt{2}t}{2\sqrt{2}}e^{-\sqrt{2}k_4\varepsilon},$$

$$\frac{\partial\tilde{x}}{\partial\varepsilon} = 2k_4\tilde{t} \implies \tilde{x} = \frac{\sqrt{2}t + x}{2}e^{\sqrt{2}k_4\varepsilon} - \frac{x - \sqrt{2}t}{2}e^{-\sqrt{2}k_4\varepsilon},$$

$$\frac{\partial\tilde{\varrho}}{\partial\varepsilon} = 0 \implies \tilde{\varrho} = \varrho,\quad \frac{\partial\tilde{S}}{\partial\varepsilon} = 0 \implies \tilde{S} = S.$$

The investigation of this algebra using *MathLie* [3] gives the following results. The commutator table is given by:

$[v_i, v_j]$	v_1	v_2	v_3	v_4	v_5
v_1	0	0	0	0	0
v_2	0	0	0	$2v_2$	0
v_3	0	0	0	v_3	0
v_4	0	$-2v_2$	$-v_3$	0	0
v_5	0	0	0	0	0

The non–trivial algebra elements are $\{v_1, v_2, v_3, v_4, v_5\}$ and the algebra generating elements reads $\{v_1, v_2, v_3, v_4, v_5\}$. Also the Cartan matrix is given by:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Furthermore, we see that the algebra is not semisimple and not nilpotent, but it is solvable. We find the following subalgebras: A subalgebra with zero elements:$\{\ \}$; subalgebras with one element: $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$; subalgebras with two elements: $\{v_1,v_2\}$, $\{v_1,v_3\}$, $\{v_1,v_4\}$, $\{v_1,v_5\}$, $\{v_2,v_3\}$, $\{v_2,v_4\}$, $\{v_2,v_5\}$, $\{v_3,v_4\}$, $\{v_3,v_5\}$, $\{v_4,v_5\}$; subalgebras with three elements: $\{v_1,v_2,v_3\}$, $\{v_1,v_2,v_4\}$, $\{v_1,v_2,v_5\}$, $\{v_1,v_3,v_4\}$, $\{v_1,v_3,v_5\}$, $\{v_1,v_4,v_5\}$, $\{v_2,v_3,v_4\}$, $\{v_2,v_3,v_5\}$, $\{v_2,v_4,v_5\}$, $\{v_3,v_4,v_5\}$; subalgebras with four elements: $\{v_1,v_2,v_3,v_4\}$, $\{v_1,v_2,v_3,v_5\}$, $\{v_1,v_2,v_4,v_5\}$, $\{v_1,v_3,v_4,v_5\}$, $\{v_2,v_3,v_4,v_5\}$; subalgebra with five elements: $\{v_1,v_2,v_3,v_4,v_5\}$. Ideals of the algebra A_1 are: $\{\ \}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_5\}$, $\{v_1,v_2\}$, $\{v_1,v_3\}$, $\{v_1,v_5\}$, $\{v_2,v_3\}$, $\{v_2,v_5\}$, $\{v_3,v_5\}$, $\{v_1,v_2,v_3\}$, $\{v_1,v_2,v_5\}$, $\{v_1,v_3,v_5\}$, $\{v_2,v_3,v_4\}$, $\{v_2,v_3,v_5\}$, $\{v_1,v_2,v_3,v_4\}$, $\{v_1,v_2,v_3,v_5\}$, $\{v_2,v_3,v_4,v_5\}$, $\{v_1,v_2,v_3,v_4,v_5\}$. The radical is $\{v_1,v_2,v_3,v_4,v_5\}$. The center of the algebra is $\{v_1,v_5\}$. The adjoint representation in matrix–form is given by

$$\mathrm{Ad}(\varepsilon_1 v_1)=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\qquad \mathrm{Ad}(\varepsilon_2 v_2)=\begin{pmatrix}1&0&0&0&0\\0&1&0&-2\varepsilon_2&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_3 v_3)=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&-\varepsilon_3&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\qquad \mathrm{Ad}(\varepsilon_4 v_4)=\begin{pmatrix}1&0&0&0&0\\0&e^{2\varepsilon_4}&0&0&0\\0&0&e^{\varepsilon_4}&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_5 v_5)=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The second algebra A_2 is

$$v_1=\partial_t\cdot,\quad v_2=\partial_x\cdot,\quad v_3=2t\partial_t\cdot+x\partial_x\cdot,\quad v_4=\partial_S\cdot,\quad v_5=x\partial_S\cdot+t\partial_x\cdot,$$
$$v_6=x^2\partial_S\cdot+2t^2\partial_t\cdot+2tx\partial_x\cdot-2t\varrho\partial_\varrho\cdot,\quad v_7=\varrho\partial_\varrho\cdot$$

with the commutator table

$[v_i,v_j]$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	0	$2v_1$	0	v_2	$2v_3-2v_7$	0
v_2	0	0	v_2	0	v_4	$2v_5$	0
v_3	$-2v_1$	$-v_2$	0	0	v_5	$2v_6$	0
v_4	0	0	0	0	0	0	0
v_5	$-v_2$	$-v_4$	$-v_5$	0	0	0	0
v_6	$-2v_3+2v_7$	$-2v_5$	$-2v_6$	0	0	0	0
v_7	0	0	0	0	0	0	0

The non–trivial algebra elements are $\{v_1,v_6\}$ and the algebra generating elements read $\{v_1,v_2,v_3,v_6\}$, $\{v_1,v_2,v_6,v_7\}$, $\{v_1,v_3,v_5,v_6\}$, $\{v_1,v_5,v_6,v_7\}$, $\{v_1,v_2,v_3,v_4,v_6\}$, $\{v_1,v_2,v_3,v_5,v_6\}$, $\{v_1,v_2,v_3,v_6,v_7\}$, $\{v_1,v_2,v_4,v_6,v_7\}$, $\{v_1,v_2,v_5,v_6,v_7\}$, $\{v_1,v_3,v_4,v_5,v_6\}$, $\{v_1,v_3,v_5,v_6,v_7\}$,

$\{v_1, v_4, v_5, v_6, v_7\}$, $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, $\{v_1, v_2, v_3, v_4, v_6, v_7\}$, $\{v_1, v_2, v_3, v_5, v_6, v_7\}$, $\{v_1, v_2, v_4, v_5, v_6, v_7\}$, $\{v_1, v_3, v_4, v_5, v_6, v_7\}$, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. The Cartan matrix can be calculated as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, the algebra A_2 is not semisimple, not solvable and not nilpotent. The following subalgebras can be calculated: A subalgebra with zero element $\{\ \}$; subalgebras with one element: $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$, $\{v_6\}$, $\{v_7\}$; subalgebras with two elements: $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_1, v_7\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_2, v_7\}$, $\{v_3, v_4\}$, $\{v_3, v_5\}$, $\{v_3, v_6\}$, $\{v_3, v_7\}$, $\{v_4, v_5\}$, $\{v_4, v_6\}$, $\{v_4, v_7\}$, $\{v_5, v_6\}$, $\{v_5, v_7\}$, $\{v_6, v_7\}$; subalgebras with three elements: $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_2, v_7\}$, $\{v_1, v_3, v_4\}$, $\{v_1, v_3, v_7\}$, $\{v_1, v_4, v_7\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_3, v_7\}$, $\{v_2, v_4, v_5\}$, $\{v_2, v_4, v_7\}$, $\{v_3, v_4, v_5\}$, $\{v_3, v_4, v_6\}$, $\{v_3, v_4, v_7\}$, $\{v_3, v_5, v_6\}$, $\{v_3, v_5, v_7\}$, $\{v_3, v_6, v_7\}$, $\{v_4, v_5, v_6\}$, $\{v_4, v_5, v_7\}$, $\{v_4, v_6, v_7\}$, $\{v_5, v_6, v_7\}$; subalgebras with four elements: $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_7\}$, $\{v_1, v_2, v_4, v_5\}$, $\{v_1, v_2, v_4, v_7\}$, $\{v_1, v_3, v_4, v_7\}$, $\{v_1, v_3, v_6, v_7\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_2, v_3, v_4, v_7\}$, $\{v_2, v_4, v_5, v_6\}$, $\{v_2, v_4, v_5, v_7\}$, $\{v_3, v_4, v_5, v_6\}$, $\{v_3, v_4, v_5, v_7\}$, $\{v_3, v_4, v_6, v_7\}$, $\{v_3, v_5, v_6, v_7\}$, $\{v_4, v_5, v_6, v_7\}$; subalgebras with five elements: $\{v_1, v_2, v_3, v_4, v_5\}$, $\{v_1, v_2, v_3, v_4, v_7\}$, $\{v_1, v_2, v_4, v_5, v_7\}$, $\{v_1, v_3, v_4, v_6, v_7\}$, $\{v_2, v_3, v_4, v_5, v_6\}$, $\{v_2, v_3, v_4, v_5, v_7\}$, $\{v_2, v_4, v_5, v_6, v_7\}$, $\{v_3, v_4, v_5, v_6, v_7\}$; subalgebras with six elements: $\{v_1, v_2, v_3, v_4, v_5, v_7\}$, $\{v_2, v_3, v_4, v_5, v_6, v_7\}$; subalgebra with seven elements: $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

The algebra–ideals are $\{\ \}$, $\{v_4\}$, $\{v_7\}$, $\{v_4, v_7\}$, $\{v_2, v_4, v_5\}$, $\{v_2, v_4, v_5, v_7\}$, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and the algebra–radical is $\{v_2, v_4, v_5, v_7\}$. The algebra–center reads $\{v_4, v_7\}$. The adjoint representation of the algebra is

$$\mathrm{Ad}(\varepsilon_1 v_1) = \begin{pmatrix} 1 & 0 & -2\varepsilon_1 & 0 & 0 & 2\varepsilon_1^2 & 0 \\ 0 & 1 & 0 & 0 & -\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2\varepsilon_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\varepsilon_1 & 1 \end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_2 v_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\varepsilon_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\varepsilon_2 & \varepsilon_2^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2\varepsilon_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_3 v_3) = \begin{pmatrix} e^{2\varepsilon_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\varepsilon_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\varepsilon_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_4 v_4) = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \qquad \mathrm{Ad}(\varepsilon_5 v_5) = \begin{pmatrix} 1&0&0&0&0&0&0\\ \varepsilon_5&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ \frac{1}{2}\varepsilon_5^2&\varepsilon_5&0&1&0&0&0\\ 0&0&\varepsilon_5&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_6 v_6) = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 2\varepsilon_6&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&2\varepsilon_6&0&0&1&0&0\\ 2\varepsilon_6^2&0&2\varepsilon_6&0&0&1&0\\ 2\varepsilon_6&0&0&0&0&0&1 \end{pmatrix}, \qquad \mathrm{Ad}(\varepsilon_7 v_7) = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}.$$

3 Optimal systems

We consider a system of differential equations with a r-parametric symmetry group

$$F\left(\vec{x},\vec{u},\vec{u}^{(r)}\right) = \left(F_1\left(\vec{x},\vec{u},\vec{u}^{(r)}\right),\ldots,F_m\left(\vec{x},\vec{u},\vec{u}^{(r)}\right)\right).$$

For every s-paramteric subgroup H we can find under the assumption that $s < \min(r, n') - n' =$ number of the independent variables $-$ a family of similarity solutions. It is impossible to calculate all kinds of similarity solutions because there are infinitely many such subgroups.

In this set of similarity solutions there are such solutions, which can be calculated by transformation of the symmetry group from other similarity solutions. The aim is to calculate a minimal set of similarity solutions from which one can gain all the other similarity solutions by transformation. Such a list is called optimal system and the elements are essentially different similarity solutions [7, 8, 9]. With group theoretical and algebraical considerations we can transform this problem to that of classifying the Lie subalgebras [9, 10]. The tools to do this are the Campbell–Baker–Hausdorff formula and the adjoint representation of the Lie algebra. These tools are now applied to the algebra A_1. The general adjoint representation which can be calculated by the matrix product of all adjoint representations of A_1 is

$$\mathrm{Ad}_g = \begin{pmatrix} 1&0&0&0&0\\ 0&e^{2\varepsilon_4}&0&-2\varepsilon_2 e^{2\varepsilon_4}&0\\ 0&0&e^{\varepsilon_4}&-3\varepsilon_3 e^{\varepsilon_4}&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}.$$

We have to simplify the following system of equations:

$$\frac{1}{a}\mathrm{Ad}_g \cdot \vec{\alpha} = \vec{\beta} \qquad \text{with} \quad \vec{\alpha} = (\alpha_1,\ldots,\alpha_5)^T \quad \text{and} \quad \vec{\beta} = (\beta_1,\ldots,\beta_5)^T.$$

This is the system

$$\frac{1}{a}\alpha_1 = \beta_1, \tag{6}$$

$$\frac{1}{a}\left(e^{2\varepsilon_4}\alpha_2 - 2\varepsilon_2 e^{2\varepsilon_4}\alpha_4\right) = \beta_2, \tag{7}$$

$$\frac{1}{a}\left(e^{\varepsilon_4}\alpha_3 - 3\varepsilon_3 e^{\varepsilon_4}\alpha_4\right) = \beta_3, \tag{8}$$

$$\frac{1}{a}\alpha_4 = \beta_4, \tag{9}$$

$$\frac{1}{a}\alpha_5 = \beta_5. \tag{10}$$

Special cases:

- $\alpha_4 \neq 0$. From equation (7), (8) follows that $\varepsilon_2 = \frac{\alpha_2}{2\alpha_4}$, $\varepsilon_3 = \frac{\alpha_3}{3\alpha_4}$. Therefore we have $\vec{\beta} = (1, 0, 0, 1, 1)$;
- $\alpha_4 = 0$, $\alpha_2 \neq 0$. From equation (7) and (8) we have $\frac{1}{a}\left(e^{2\varepsilon_4}\alpha_2\right) = \pm 1$. It follows that $e^{\varepsilon_4} = \sqrt{\frac{1}{|\alpha_2|}}$. Furthermore it is $\frac{1}{a}\left(e^{\varepsilon_4}\alpha_3\right) = \pm 1$ from which we get $\pm\frac{1}{a}\sqrt{\frac{1}{|\alpha_2|}} = \pm 1$. Therefore $\vec{\beta}$ is $\vec{\beta} = (1, \pm 1, \pm 1, 0, 1)$. This gives four linearly independent vectors: $\vec{\beta}_1 = (1, 1, 1, 0, 1)$, $\vec{\beta}_2 = (1, 1, -1, 0, 1)$, $\vec{\beta}_3 = (1, -1, 1, 0, 1)$, $\vec{\beta}_4 = (1, -1, -1, 0, 1)$;
- $\alpha_4 = 0$, $\alpha_2 = 0$, $\alpha_3 \neq 0$. From equation (8) we have $\frac{1}{a}\left(e^{\varepsilon_4}\alpha_3\right) = \pm 1$ from which follows $e^{\varepsilon_4} = \pm\frac{1}{\alpha_3}$. In this case $\vec{\beta} = (1, 0, \pm 1, 0, 1)$. This gives two linearly independent vectors: $\vec{\beta}_1 = (1, 0, 1, 0, 1)$, $\vec{\beta}_2 = (1, 0, -1, 0, 1)$;
- $\alpha_4 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$. Therefore $\vec{\beta}$ is $\vec{\beta} = (1, 0, 0, 0, 1)$;
- $\alpha_4 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_1 = 0$. Therefore $\vec{\beta}$ is $\vec{\beta} = (0, 0, 0, 0, 1)$.

4 Solutions of the Doebner–Goldin–Madelung equations

During the last part of this paper we want to show how to construct solutions for special subalgebras. To do this we choose for example the algebra A_2. Here we consider the generators v_5 and v_6 which generates an Abelian group. The algorithm for calculating solutions is represented in the literature [3, 11]. Because of the fact that the choosen subalgebra is Abelian we have to solve the following equations:

$$v_5 I = 0, \qquad v_6 I = 0,$$

where I are the invariants. We start with the second equation containing v_6. The connected system of characteristics results:

$$\frac{\partial x}{\partial s} = 2t(s)x(s), \qquad \frac{\partial t}{\partial s} = 2t(s)^2, \qquad \frac{\partial \varrho}{\partial s} = -2t(s)\varrho(s), \qquad \frac{\partial S}{\partial s} = x(s)^2. \tag{11}$$

This system can be solved by repeated isolation of s from the other variables of the system (11). The solution is

$$t = \frac{1}{-2s - C_1}, \qquad x = \frac{C_2}{2s + C_1}, \qquad \varrho = (2s + C_1)C_3, \qquad S = \frac{-C_2^2}{2(2s + C_1)} + C_4. \tag{12}$$

Here C_1, C_2, C_3, C_4 are constants of integration. They play the role of the invariants. To eliminate the parameter s we solve the first equation of (12) with respect to the parameter s and put the result into the other equations. We then get

$$s = \frac{1 + tC_1}{-2t}, \qquad x = -tC_2, \qquad \varrho = -\frac{C_3}{t}, \qquad S = \frac{1}{2}\frac{x^2}{t} + C_4.$$

Therefore the invariants are received by isolating the constants:

$$I_1 = \frac{x}{t} = -C_2, \qquad I_2 = t\varrho = -C_3, \qquad I_3 = S - \frac{1}{2}\frac{x^2}{t} = C_4.$$

The next step is to transform the generator v_5 into the basis of the invariants. This means that it is of the form

$$v_5 = (v_5 I_1)\frac{\partial\cdot}{\partial I_1} + (v_5 I_2)\frac{\partial\cdot}{\partial I_2} + (v_5 I_3)\frac{\partial\cdot}{\partial I_3}.$$

Here, the result simply is $v_5 = \frac{\partial\cdot}{\partial I_1}$. Now we have to calculate the invariants from this generator. The system of characteristic equations is

$$\frac{\partial I_1}{\partial s} = 1, \qquad \frac{\partial I_2}{\partial s} = 0, \qquad \frac{\partial I_3}{\partial s} = 0,$$

with the solution

$$I_1 = s + C_1, \qquad J_2 = I_2 = t\varrho, \qquad J_3 = I_3 = S - \frac{1}{2}\frac{x^2}{t},$$

where $C_1 = J_1$, J_2 and J_3 are the invariants of this generator. Now it is

$$J_2 = \phi(J_1), \qquad J_3 = \psi(J_1).$$

With $s = 0$ this leads to

$$\varrho = \frac{\phi(\alpha)}{t}, \qquad S = \psi(\alpha) + \frac{x^2}{2t}, \qquad \alpha = \frac{x}{t}. \tag{13}$$

In this case we can put $s = 0$ because of the fact that we freely can choose the origin of the coordinate system. For calculating the solution we choose equation 2 from Table 1. By introducing the expressions for ϱ and S from the equations (13) we find the following ordinary differential equations:

$$\phi_\alpha(\alpha)\psi_\alpha(\alpha) + \phi(\alpha)\psi_{\alpha\alpha}(\alpha) = 0,$$
$$\phi_\alpha(\alpha)^2 + 2\phi(\alpha)\big(2\phi(\alpha)\psi_\alpha(\alpha)^2 + (4D_2 - 1)\phi_{\alpha\alpha}(\alpha)\big) = 0.$$

It is easy to integrate the first equation to $\psi_\alpha(\alpha) = \frac{A}{\phi(\alpha)}$. By introducing this into the second equation we find

$$4A^2 + \phi_\alpha(\alpha)^2 + 2(4D_2 - 1)\phi(\alpha)\phi_{\alpha\alpha}(\alpha) = 0.$$

To solve this equation, we first divide by the coefficient in front of the second derivative. This provides

$$\phi_{\alpha\alpha}(\alpha) + \frac{\phi_\alpha(\alpha)^2}{2(4D_2 - 1)\phi(\alpha)} + \frac{4A^2}{2(4D_2 - 1)\phi(\alpha)} = 0.$$

Now we substitute $p(\phi) = \phi_\alpha(\alpha)$. The result is

$$pp_\phi + \frac{p^2}{2(4D_2 - 1)\phi} + \frac{4A^2}{2(4D_2 - 1)\phi} = 0$$

with the solution:

$$p = \pm\sqrt{C[1]^2\phi(\alpha)^{\frac{1}{1-4D_2}} - 4A^2}.$$

In the next step we have to calculate $p(\phi) = \phi_\alpha(\alpha)$. With $D_2 = \frac{1}{8}$ this leads to the solution:

$$\phi(\alpha) = \frac{4A^2 e^{\pm\alpha C[1] - C[2]}}{C[1]} + \frac{e^{\mp\alpha C[1] + C[2]}}{4C[1]}.$$

For the other function we find :

$$\psi(\alpha) = B + \arctan\left((4A)^{\pm 1} e^{\alpha C[1] \mp C[2]}\right).$$

This leads to

$$\varrho(x,t) = \frac{4A^2 e^{\pm \frac{x}{t} C[1] - C[2]}}{C[1]t} + \frac{e^{\mp \frac{x}{t} C[1] + C[2]}}{4C[1]t},$$
$$S(x,t) = B + \frac{x^2}{2t} + \arctan\left((4A)^{\pm 1} e^{\frac{x}{t} C[1] \mp C[2]}\right). \tag{14}$$

To prove that these are solutions of the original equations we put these expressions into system 2 of Table 1. The result is valid with $D_2 = \frac{1}{8}$. The original form of the Doebner–Goldin equation then can be solved by a $\psi(x,t)$ according to the Madelung transformation (3) in $1+1$ dimensions. Especially when discussing a time–independent potential $V(x,t) = \delta(x-a)$ with natural boundary conditions, the results (14) can be useful.

Another special case is given with $D_2 = \frac{1}{4}$ leading to the following differential equation:

$$4A^2 + \phi_\alpha(\alpha)^2 = 0.$$

To get a real–valued density $\varrho(x,t)$ out of this system, $A = i\aleph$ is an imaginary integration constant leading to the following result which is valid formally for any D_2, but for $A = \aleph = 0$ and no imaginary rest in the original Schrödinger context only:

$$\varrho(x,t) = \frac{C[1]}{t} - \frac{2\aleph x}{t^2}, \qquad S(x,t) = B + \frac{x^2}{2t} - \frac{i}{2}\ln\left(\frac{2\aleph x}{t} - C[1]\right).$$

The next case is $A = 0$. The connected differential equation simply is

$$\phi_\alpha^2 + 2(4D_2 - 1)\phi\phi_{\alpha\alpha} = 0$$

with the obvious solution

$$\phi(\alpha) = \left(\frac{(\pm\alpha C[1] - C[2])\,(8D_2 - 1)}{8D_2 - 2}\right)^{\frac{8D_2-2}{8D_2-1}}.$$

Therefore the solution of the original system is

$$\varrho(x,t) = \frac{\left(\frac{\left(\pm\frac{x}{t}C[1] - C[2]\right)(8D_2-1)}{8D_2-2}\right)^{\frac{8D_2-2}{8D_2-1}}}{t}, \qquad S(x,t) = B + \frac{x^2}{2t},$$

where B, $C[1]$, $C[2]$ are constants. We note that the special case $D_2 = \frac{1}{8}$ leads to a different solution of type (14).

5 Conclusion

This paper demonstrates the solution of the system of the Doebner–Goldin–Madelung equations. The main tool for solving this system was the computer algebra package *MathLie* written in *Mathematica*. By applying this tool to equation (5) we have derived analytical solutions. We demonstrated that *MathLie* is also able to examine the Lie group and the related Lie algebra.

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