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A Novel Nonlinear Evolution Equation Integrable by the Inverse Scattering Method

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In this report we consider the nonlinear evolution equation $(u_t + uu_x)_x + u = 0$ (Vakhnenko equation – VE) that can be integrated by the inverse scattering transform (IST) method. This equation arose as a result describing the high-frequency perturbations in a relaxing medium. The VE has two families of travelling wave solutions, both of which are stable to long wavelength perturbations. In particular, the VE has a loop-like soliton solution. The interaction of two solitons by both Hirota's method and the IST method are considered. The associated eigenvalue problem has been formulated. This has been achieved by finding a Bäcklund transformation. The inverse scattering method has a third order eigenvalue problem. Under the interaction of solitons there are features that are not typical for the KdV equation.

1 Introduction

Describing real media under the action of intense waves is often unsuccessful in the framework of equilibrium models of continuum mechanics. To develop physical models for wave propagation through media with complicated inner kinetics, the notions based on the relaxational nature of a phenomenon are regarded to be promising. A nonlinear evolution equation is suggested to describe the propagation of waves in a relaxing medium [1]. It is shown that for the low-frequency approach this equation is reduced to the Korteweg-de Vries (KdV) equation. In contrast to the low-frequency perturbations, the high-frequency perturbations satisfy a new nonlinear equation [2]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0. \tag{1}$$

The equation (1) has been studied in various Refs. [2, 3, 4, 5, 6, 7, 8, 9]. Hereafter, as was initiated in [3], this equation is referred to as the Vakhnenko equation (VE). There is a certain analogy between the KdV equation and the VE. They have the same hydrodynamic nonlinearity and do not contain dissipative terms; only the dispersive terms are different. It turns out that the VE possesses, at least partially, the remarkable properties inherent to the KdV equation. The study of the VE has scientific interest both from the viewpoint of the existence of stable wave formations and from the viewpoint of the general problem of integrability of nonlinear equations.

2 Physical processes described by the Vakhnenko equation

From the nonequilibrium thermodynamics standpoint, the models of a relaxing medium are more general than the equilibrium models. Thermodynamic equilibrium is disturbed owing to

the propagation of fast perturbations. There are processes of the interaction that tend to return the equilibrium. In essence, the change of macroparameters caused by the changes of inner parameters is a relaxation process.

To analyze the wave motion, we use the hydrodynamic equations in Lagrangian coordinaties:

$$\frac{\partial V}{\partial t} - \frac{1}{\rho_0}\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0. \tag{2}$$

The following dynamic state equation is applied to account for the relaxation effects:

$$d\rho = c_f^{-2} dp + \tau_p^{-1}(\rho - \rho_e)\, dt. \tag{3}$$

We note that the mechanisms of the exchange processes are not defined concretely when deriving the dynamic state equation (3). In this equation the thermodynamic and kinetic parameters appear only as sound velocities c_e, c_f and relaxation time τ_p. These characteristics can be found experimentally.

Let us consider a small nonlinear perturbation $p' < p_0$. Combining the relationships (2), (3) we obtain the nonlinear evolution equation in one unknown p (the dash in p' is omitted) [1]

$$\tau_p \frac{\partial}{\partial t}\left(\frac{\partial^2 p}{\partial x^2} - c_f^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_f \frac{\partial^2 p^2}{\partial t^2}\right) + \left(\frac{\partial^2 p}{\partial x^2} - c_e^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_e \frac{\partial^2 p^2}{\partial t^2}\right) = 0. \tag{4}$$

A similar equation has been obtained by Clarke [10], but without nonlinear terms.

In [1] it is shown that for low-frequency perturbations ($\tau_p \omega \ll 1$) the equation (4) is reduced to the Korteweg-de Vries – Burgers (KdVB) equation

$$\frac{\partial p}{\partial t} + c_e \frac{\partial p}{\partial x} + \alpha_e c_e^3 p \frac{\partial p}{\partial x} - \beta_e \frac{\partial^2 p}{\partial x^2} + \gamma_e \frac{\partial^3 p}{\partial x^3} = 0,$$

while for high-frequency waves ($\tau_p \omega \gg 1$) we have obtained the new equation

$$\frac{\partial^2 p}{\partial x^2} - c_f^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_f c_f^2 \frac{\partial^2 p^2}{\partial x^2} + \beta_f \frac{\partial p}{\partial x} + \gamma_f p = 0. \tag{5}$$

The nonlinear equation (5) has dissipative $\beta_f \partial p/\partial x$ and dispersive $\gamma_f p$ terms. Without nonlinear and dissipative terms, we have a linear Klein–Gordon equation.

In the general case the last equation has been investigated insufficiently. It is likely that this is connected with the fact, noted by Whitham [11], that the high-frequency perturbations attenuate very fast. However in Whitham's monograph, the evolution equation without nonlinear and dispersive terms was considered. Certainly, the lack of such terms restricts the class of solutions. At least, there is no solution in the form of a solitary wave, which is caused by nonlinearity and dispersion.

3 Evolution equation for high-frequency perturbations

The equation (5), which we are interested in, is written down in dimensionless form. In the moving coordinates system with velocity c_f, the equation has the form in dimensionless variables $\tilde{x} = \sqrt{\frac{\gamma_f}{2}}(x - c_f t)$, $\tilde{t} = \sqrt{\frac{\gamma_f}{2}} c_f t$, $\tilde{u} = \alpha_f c_f^2 p$ (tilde over variables $\tilde{x}$, $\tilde{t}$, $\tilde{u}$ is omitted)

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + \alpha\frac{\partial u}{\partial x} + u = 0. \tag{6}$$

The constant $\alpha = \beta_f/\sqrt{2\gamma_f}$ is always positive. Equation (6) without the dissipative term has the form of the nonlinear equation [2, 3] (see equation (1))

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0. \tag{7}$$

The travelling-wave solutions of the VE (7) were derived in [2, 3], and its symmetry properties were studied in [5]. A remarkable feature of the VE is that it has a soliton solution which has loop-like form, i.e. it is a multi-valued function (see Fig. 1 in [2]). Whilst loop soliton solutions are rather intriguing, it is the solution to the initial value problem that is of more interest in a physical context.

The physical interpretation of the multi-valued functions that describe the loop-like soliton solutions was given in [1]. The problem is whether the ambiguity has a physical nature or is related to the incompleteness of the mathematical model, in particular to the lack of dissipation. It is significant that the loop-like solutions are stable to long-wavelength perturbations [3], and that the introduction of a dissipative term (see equation (6)), with dissipation parameter less than some limiting value, does not destroy these loop-like solutions [1]. Since the solution has a parametric form [2, 3], there is a space of variables in which the solution is a single-value function. Consequently, the ambiguity of solution does not relate to the incompleteness of the mathematical model. Thus in the framework of this model approach, the high-frequency perturbation can be described by the multi-valued functions [1].

We have succeeded in finding new coordinates (X, T), in terms of which the solution of equation (1) is given by single-valued parametric relations. New independent coordinates X, T are defined as [4]

$$x = x_0 + T + W(X,T), \qquad t = X, \qquad W = \int_{-\infty}^{X} U(X',T)\,dX'. \tag{8}$$

Here $u(x,t) = U(X,T)$, and x_0 is a constant. We also assume that, as $X \to -\infty$, the derivatives of W vanish and W tends to a constant. Equation (1) then has the form [4, 8]

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{9}$$

If the solution $U(X,T) = W_X$ of the transformed VE (9) has been obtained, the original independent space coordinate x can be found by means of the formula (8). This relationship together with $u(x,t) = U(X,T)$ enables us to define the solution of the VE in parametric form with T as parameter. We note that the transformation (9) between old and new coordinates is similar to the transformation between Eulerian variables (x,t) and Lagrangian variables (T,X) [8].

Finally, by taking $W = 6(\ln f)_X$, where f is a function of X and T, we observe that the transformed VE (9) may be written as the bilinear equation [4]

$$\left(D_T D_X^3 + D_X^2\right) f \cdot f = 0, \tag{10}$$

where D is the Hirota binary operator [12].

4 Bäcklund transformation for the transformed Vakhnenko equation

We present a Bäcklund transformation for equation (10), following the method developed in [13]. It is well known that the Bäcklund transformation is one of the analytical tools for dealing with

soliton problems and has a close relationship to the IST method [12, 13, 14]. First we define P as follows:

$$P := \left[\left(D_T D_X^3 + D_X^2\right) f' \cdot f'\right] ff - f'f' \left[\left(D_T D_X^3 + D_X^2\right) f \cdot f\right].$$

We aim to find a pair of equations such that each equation is linear in each of the dependent variables f and f', and such that together f and f' satisfy $P = 0$. The pair of equations is the required Bäcklund transformation.

Combining this relationship we can rewrite P in the following form [8]:

$$P = 2D_T\left(\left\{D_X^3 - \lambda(X)\right\} f' \cdot f\right) \cdot (f'f) - 2D_X(\{3D_T D_X + 1 + \mu(T)D_X\}f' \cdot f) \cdot (D_X f' \cdot f).$$

Thus we have proved [8] that the Bäcklund transformation is given by the two equations

$$\left(D_X^3 - \lambda\right) f' \cdot f = 0, \tag{11}$$

$$(3D_X D_T + 1 + \mu D_X)\, f' \cdot f = 0, \tag{12}$$

where $\lambda = \lambda(X)$ is an arbitrary function of X and $\mu = \mu(T)$ is an arbitrary function of T. In original form with $\mu = 0$ we have

$$(W' - W)_{XX} + \frac{1}{2}(W' - W)(W' + W)_X + \frac{1}{36}(W' - W)^3 - 6\lambda = 0, \tag{13}$$

$$(W' - W)\left[3(W' + W)_{XT} + \frac{1}{2}(W' - W)(W' - W)_T\right] - 6(W' - W)_X\left[1 + \frac{1}{2}(W' + W)_T\right] = 0. \tag{14}$$

Separately the two equations (11), (12) appear as part of the Bäcklund transformation for other nonlinear evolution equations. For example, equation (11) is the same as one of the equations that is part of the Bäcklund transformation for a higher order KdV equation (see equation (5.139) in [12]), and equation (12) is similar to (5.132) in [12] that is part of the Bäcklund transformation for a model equation for shallow water waves [15].

5 Interaction of the solitons

The transformation into new coordinates (8) is the key to solving the problem of the interaction of the solitons. The exact N-soliton solutions are obtained by use of (i) Hirota's method [4, 7]; (ii) elements of the inverse scattering transform procedure for the KdV equation (spectral equation of second order – Schrödinger equation) [6]; (iii) the inverse scattering transform procedure (spectral equation of third order) [9].

Since the equation (1) can be written in bilinear form (10), Hirota's method enables us to find soliton solutions. These solutions have been obtained in [4, 7], for example, for the one-soliton solution

$$f = 1 + \exp(2\eta), \qquad W = 6(\ln f)_X, \qquad \eta = kX - \omega T + \alpha, \qquad U = W_X = 6k^2 \operatorname{sech}^2\eta,$$

and for the two-soliton solution

$$f = 1 + \exp(2\eta_1) + \exp(2\eta_2) + b^2 \exp(2\eta_1 + 2\eta_2), \qquad W = 6(\ln f)_X,$$

$$b^2 = \frac{F\left[2(k_1 - k_2), -2(\omega_1 - \omega_2)\right]}{F\left[2(k_1 + k_2), -2(\omega_1 + \omega_2)\right]} = \frac{(k_2 - k_1)^2}{(k_2 + k_1)^2}\frac{k_1^2 + k_2^2 - k_1 k_2}{k_1^2 + k_2^2 + k_1 k_2},$$

$$\eta_i = k_i X - \omega_i T + \alpha_i, \qquad F(D_X, D_T) := D_T D_X^3 + D_X^2.$$

Now we present IST method for finding the solution of the VE. The IST is the most appropriate way of tackling the initial value problem. The results of applying the IST method would be useful in solving the Cauchy problem for the VE. In order to use the IST method one first has to formulate the associated eigenvalue problem.

Introducing the function $\psi = f'/f$, we find that equations (11), (12) reduce to

$$\psi_{XXX} + U\psi_X - \lambda\psi = 0, \tag{15}$$
$$3\psi_{XT} + (W_T + 1)\psi + \mu\psi_X = 0, \tag{16}$$

respectively. It may be shown

$$[W_{XXT} + (1 + W_T)W_X]_X\psi + \lambda_X(3\psi_T + \mu\psi) = 0.$$

Hence equation (9) is the condition for $\lambda_X = 0$, and hence for λ to be constant. Constant λ (spectral parameter) is what is required in the IST problem.

Thus the IST problem is directly related to a spectral equation of third order (15). The third order eigenvalue problem is similar to the one associated with a higher order KdV equation [16, 17], a Boussinesq equation [16, 18], and a model equation for shallow water waves [12, 19].

Kaup [16], Caudrey [18, 20] and Deift et al. [21] studied the inverse problem for certain third order spectral equations. We adapt the results obtained by these authors to the present problem and describe a procedure for using the IST to find the N-soliton solution to the transformed VE, and hence to the VE itself.

We proved that the T-evolution of the scattering data is given by the relationships [9] ($k = 1, 2, \ldots, 2N$)

$$\zeta_j^{(k)}(T) = \zeta_j^{(k)}(0),$$
$$\gamma_{1j}^{(k)}(T) = \gamma_{1j}^{(k)}(0)\exp\left\{\left[-\left(3\lambda_j\left(\zeta_1^{(k)}\right)\right)^{-1} + \left(3\lambda_1\left(\zeta_1^{(k)}\right)\right)^{-1}\right]T\right\}. \tag{17}$$

Here $\lambda_j(\zeta) = \omega_j\zeta$, $\lambda_j^3(\zeta) = \lambda$, and $\omega_j = e^{\mathrm{i}2\pi(j-1)/3}$ are the cube of roots of 1.

The final result for the N-soliton solution of the transformed VE is given by the relation [9]

$$U(X,T) = 3\frac{\partial^2}{\partial X^2}\ln\left(\det M(X,T)\right), \tag{18}$$

where M is the $2N\times 2N$ matrix given by

$$M_{kl} = \delta_{kl} - \sum_{j=2}^{3}\gamma_{1j}^{(k)}(0)\,\frac{\exp\left\{\left[-\left(3\lambda_j\left(\zeta_1^{(k)}\right)\right)^{-1} + \left(3\lambda_1\left(\zeta_1^{(k)}\right)\right)^{-1}\right]T + \left(\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(l)}\right)\right)X\right\}}{\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(l)}\right)}, \tag{19}$$

and the scattering data is calculated from constants ξ_m, β_m as

$$n = 1,\ 2, \ldots,\ N, \qquad m = 2n - 1,$$
$$\lambda_1\left(\zeta_1^{(m)}\right) = \mathrm{i}\omega_2\xi_m, \qquad \lambda_2\left(\zeta_1^{(m)}\right) = \mathrm{i}\omega_3\xi_m, \qquad \gamma_{12}^{(m)}(0) = \omega_2\beta_m, \qquad \gamma_{13}^{(m)}(0) = 0,$$
$$\lambda_1\left(\zeta_1^{(m+1)}\right) = -\mathrm{i}\omega_3\xi_m, \qquad \lambda_3\left(\zeta_1^{(m+1)}\right) = -\mathrm{i}\omega_2\xi_m, \qquad \gamma_{12}^{(m+1)}(0) = 0, \qquad \gamma_{13}^{(m+1)}(0) = \omega_3\beta_m.$$

For the N-soliton solution there are N arbitrary constants ξ_m and N arbitrary constants β_m.

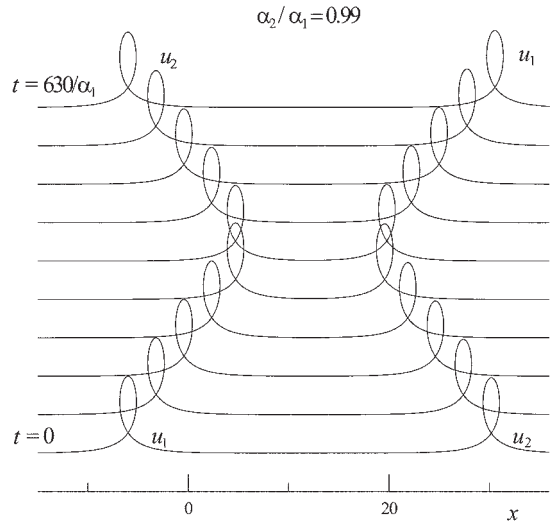

Figure 1. Interaction of two solitons in moving coordinates at time interval $\Delta t = 70/\alpha_1$.

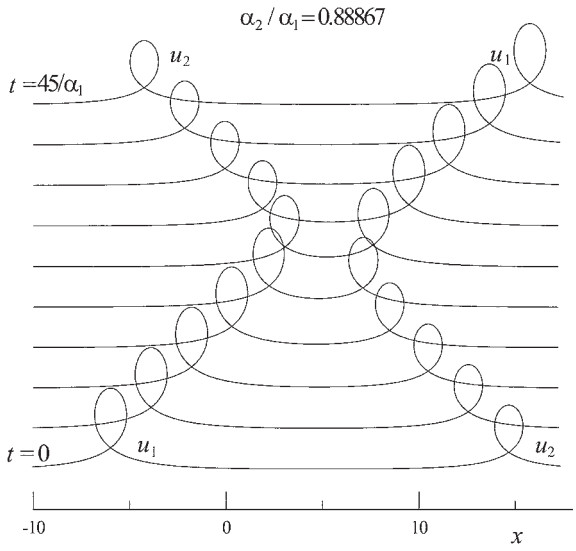

Figure 2. The phaseshifts of the smaller soliton is zero. Time interval is $\Delta t = 5/\alpha_1$.

For example, the matrix for one-soliton solution has a form

$$\begin{pmatrix} 1 - \dfrac{\omega_2\beta_1}{\sqrt{3}\,\xi_1}\exp\left[\sqrt{3}\,\xi_1 X - (\sqrt{3}\,\xi_1)^{-1}T\right] & \dfrac{\mathrm{i}\omega_3\beta_1}{2\xi_1}\exp\left[2\mathrm{i}\omega_3\xi_1 X - (\sqrt{3}\,\xi_1)^{-1}T\right] \\ \dfrac{-\mathrm{i}\omega_2\beta_1}{2\xi_1}\exp\left[-2\mathrm{i}\omega_2\xi_1 X - (\sqrt{3}\,\xi_1)^{-1}T\right] & 1 - \dfrac{\omega_3\beta_1}{\sqrt{3}\,\xi_1}\exp\left[\sqrt{3}\,\xi_1 X - (\sqrt{3}\,\xi_1)^{-1}T\right] \end{pmatrix}. \quad (20)$$

Calculating the determinant

$$\det M = \left\{1 + \frac{\beta_1}{2\sqrt{3}\,\xi_1}\,\exp\left[\sqrt{3}\,\xi_1\left(X - \frac{T}{3\xi_1^2}\right)\right]\right\}^2,$$

we have from (18) the one-soliton solution of the transformed VE as obtained by the IST method

$$U = 3\frac{\partial^2}{\partial X^2}\ln\left(\det M(X,T)\right) = \frac{9}{2}\,\xi_1^2\,\mathrm{sech}^2\left[\frac{\sqrt{3}}{2}\,\xi_1\left(X - \frac{T}{3\xi_1^2}\right) + \alpha_1\right], \quad (21)$$

where $\alpha_1 = \frac{1}{2}\ln(\beta_1/2\sqrt{3}\xi_1)$ is an arbitrary constant.

The determinant of the matrix for two-soliton solution has a form

$$\det M = \left(1 + q_1^2 + q_2^2 + b^2 q_1^2 q_2^2\right)^2, \quad (22)$$

where

$$q_i = \exp\left[\frac{\sqrt{3}}{2}\,\xi_i\left(X - \frac{T}{3\xi_i^2}\right) + \alpha_i\right], \qquad b^2 = \left(\frac{\xi_2 - \xi_1}{\xi_2 + \xi_1}\right)^2 \frac{\xi_1^2 + \xi_2^2 - \xi_1\xi_2}{\xi_1^2 + \xi_2^2 + \xi_1\xi_2},$$

and $\alpha_i = \frac{1}{2}\ln(\beta_i/2\sqrt{3}\xi_i)$ are arbitrary constants.

In the interaction of two solitons for the VE [4, 7, 6] there are features that are not typical for the KdV equation (see Figs. 1–3). The larger soliton moving with larger velocity catches up with the smaller soliton moving in the same direction. For convenience in the figures, the interactions of solitons are shown in coordinates moving with the speed of the centre mass.

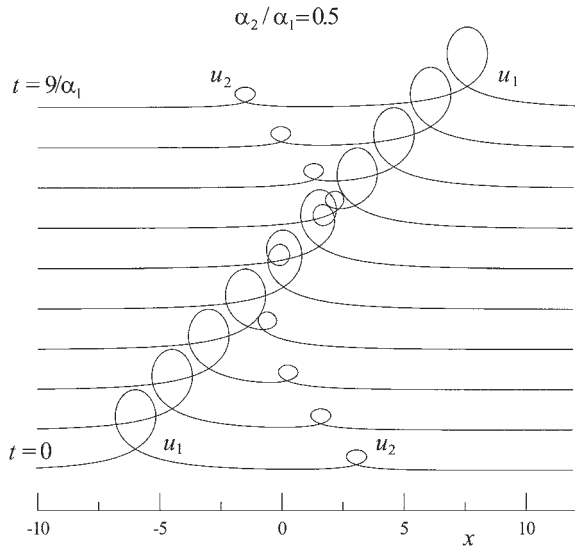

Figure 3. Both solitons have phaseshifts in the same direction. Time interval is $\Delta t = 1/\alpha_1$.

After the nonlinear interaction the solitons separate, their forms are restored, but phaseshifts arise. The larger soliton always has a forward phaseshift, while the smaller soliton can have three kinds of phaseshift. Note that this property is not typical for the KdV equation. There is a special value of the ratio $(\alpha_2/\alpha_1)^* = 0.88867$. The different kinds of phaseshift are illustrated in Figs. 1–3.

- For $\alpha_2/\alpha_1 > (\alpha_2/\alpha_1)^*$ the phaseshift of smaller soliton is in the opposite direction to the phaseshift of the larger soliton (Fig. 1).
- For $\alpha_2/\alpha_1 = (\alpha_2/\alpha_1)^*$ the smaller soliton has no phaseshift (Fig. 2).
- For $\alpha_2/\alpha_1 < (\alpha_2/\alpha_1)^*$ less critical value both solitons have phaseshifts in the same direction (Fig. 3).

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