

The Most Symmetric Drift Waves

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Comparative symmetry analysis is done for Korteweg-de Vries and Hasegawa–Mima models, both continuous Lie symmetries and discrete ones are taken into account. The form of the most symmetrical smooth solutions is determined for the Hasegawa–Mima model.

1 Introduction

Low frequency drift oscillations play an important role in the transport processes in magnetized plasmas, so they are intensely studied in recent decades [1]. The main problem in the drift waves investigations is the presence of nonlinear effects even at relatively small amplitudes. Nonlinear generation of the high space harmonics and their accumulation in the initial disturbance zone complicate numerical simulations of the drift waves evolution [2]. Thus some analytical approach based on symmetry analysis of the model is needed.

In the present work, comparative symmetry analysis is carried out for Hasegawa–Mima model for the drift waves in a plasma and for the well known Korteweg–de Vries (KdV) model. In the Section 2, symmetries and the most symmetric solutions of the KdV model are reviewed as an illustrative example. This model has sufficiently large symmetry for the existence of a family of the most symmetric stable solutions called solitons. In the Section 3, Hasegawa–Mima model symmetries and solutions are considered, both continuous and discrete symmetries are taken into account. General form of the most symmetrical solutions of the Hasegawa–Mima model is determined.

2 Korteweg-de Vries model (an illustrative example)

Korteweg-de Vries model equation for nonlinear waves with potential nonlinearity (see, e.g. [3])

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

admits the following Lie group of transformations [4]:

- a) time and space shifts: $t' = t + C_1$, $x' = x + C_2$;
- b) similarity transform: $t' = t \exp(3C_3)$, $x' = x \exp(C_3)$, $u' = u \exp(-2C_3)$;
- c) Galilean transform: $x' = x + C_4 t$, $u' = u + C_4$,

where C_1, \dots, C_4 are arbitrary constants.

In addition, KdV equation admits the reflection symmetry transform $t' = -t$, $x' = -x$.

This symmetry group is large enough for the existence of a family of stable solitary wave solutions called solitons. These solutions are the most symmetric localized smooth solutions of the KdV equation. Let us review how they can be obtained by the symmetry approach.

When we introduce the homogeneous boundary condition

$$u \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty$$

the Galilean symmetry is lost, so $C_4 = 0$. Similarity transform invariant solutions are unbounded, so we must put also $C_3 = 0$. Thus, the initial symmetry is reduced to the time and

space shifts combined with the reflection transform. As a result, the most symmetric solution must be an even function of the argument $x - vt$:

$$u = u(x - vt), \quad v = \text{const}, \quad u(-x) = u(x).$$

Inserting this into the KdV equation and solving the corresponding ordinary differential equation, we obtain a family of solutions

$$u = 12a^2 \text{sech}^2(ax - 4a^3t),$$

where the similarity group orbit is labelled by an arbitrary constant a . These solutions are the well known KdV solitons. Similar considerations allow us to obtain space periodic solutions of the KdV equation known as cnoidal waves.

As for very small amplitude solutions $u = \varepsilon \exp(ikx - i\omega t)$, $|\varepsilon| \ll 1$, they obey the dispersion relation $\omega = -k^3$ and their phase and group velocities are $\frac{\omega}{k} = -k^2$ and $\frac{\partial \omega}{\partial k} = -3k^2$ respectively. So shorter waves have higher velocities and leave the initial disturbance domain faster than longer ones. For the waves with finite but small amplitude this effect compensates the nonlinear breaking of the waves. This is the physical reason for the existence of the stable soliton solutions of the KdV model.

3 Hasegawa–Mima model symmetry and solutions

Let us consider an inhomogeneous plasma slab in the external homogeneous magnetic field. Electrons, unlike ions, are magnetized, smoothing an electrostatic potential Φ along the magnetic field lines. In this case, Hasegawa–Mima model equations hold [1]:

$$\frac{\partial \Psi}{\partial t} + J(\Phi, \Psi) = \frac{\partial \Phi}{\partial y}, \quad \Psi = \Phi - \Delta_{\perp} \Phi, \quad (1)$$

where $\Psi \equiv \Psi_z$ is the generalized vorticity, $J(F, G) \equiv \frac{\partial F}{\partial x} \frac{\partial G}{\partial y} - \frac{\partial G}{\partial x} \frac{\partial F}{\partial y}$ the Jacobian nonlinear operator and $\Delta_{\perp} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Note that vortex nonlinearity term $J(\Phi, \Psi)$ in (1) is degenerate: zero value of this term means that there exists some functional dependence between the vorticity Ψ and the electrostatic potential Φ . As a consequence, monochromatic wave solutions exist

$$\Phi = \alpha \exp(ik_1x + ik_2y - i\omega t), \quad \Psi = (1 + k_1^2 + k_2^2) \Phi \quad (2)$$

satisfying the dispersion relation

$$\omega = -\frac{k_2}{1 + k_1^2 + k_2^2}. \quad (3)$$

The amplitude α can be arbitrary, not necessarily small, because of vanishing of the nonlinear term in this case. On the other hand, it is clear from (3) that short waves are slower than the long ones, so dispersion effects cannot balance the nonlinear wave breaking. All these properties are in a sharp contrast with those of the KdV model.

Now let us try to proceed in the way used for the KdV model and obtain the most symmetric solutions of the Hasegawa–Mima equations.

First, it is useful to perform the simplifying transformation

$$\Phi = \Phi(t, x, y + t) - x, \quad \Psi = \Psi(t, x, y + t) - x. \quad (4)$$

The new RHS functions Φ and Ψ of arguments $t, x, y + t$ satisfy the equations (1) without the dispersion term $\frac{\partial \Phi}{\partial y}$, but the boundary conditions become more complicated:

$$\Phi = x, \quad \Psi = x \quad \text{as} \quad |r| \rightarrow \infty, \quad r = (x^2 + y^2)^{1/2}. \quad (5)$$

The symmetry properties of the Hasegawa–Mima model look simpler in these new notations. The related symmetry group contains four translations

$$t' = t + C_1, \quad x' = x + C_2, \quad y' = y + C_3, \quad \Phi' = \Phi + C_4, \quad \Psi' = \Psi + C_4$$

the rotation around the Oz -axis

$$x' = x \cos C_5 - y \sin C_5, \quad y' = x \sin C_5 + y \cos C_5$$

and the similarity transform

$$t' = t \exp(C_6), \quad \Phi' = \Phi \exp(-C_6), \quad \Psi' = \Psi \exp(-C_6),$$

where C_1, \dots, C_6 are arbitrary constants.

There are also three reflection symmetries:

$$\begin{aligned} a) \quad & x' = -x, \quad \Phi' = -\Phi, \quad \Psi' = -\Psi; \\ b) \quad & t' = -t, \quad y' = -y; \\ c) \quad & t' = -t, \quad \Phi' = -\Phi, \quad \Psi' = -\Psi. \end{aligned}$$

Now let us return to the most symmetric solution satisfying the homogeneous boundary conditions which in our new variables have the form (5). Similarity and rotation transforms are incompatible with these conditions, so we must put $C_5 = C_6 = 0$. The only remaining translations are time shift (C_1), y -shift (C_3) and the combination of x, Φ , and Ψ (C_2) shifts:

$$t' = t + C_1, \quad x' = x + C_2, \quad \Phi' = \Phi + C_2, \quad \Psi' = \Psi + C_2, \quad y' = y + C_3,$$

and only (a) and (b) reflections remain.

Returning (by the transformation inverse to (4)) to the initial variables and the initial form (1) of the Hasegawa–Mima equations, we obtain the following symmetries compatible with homogeneous boundary conditions:

$$\begin{aligned} t' &= t + C_1, \quad x' = x + C_2, \quad y' = y + C_3, \\ a) \quad & x' = -x, \quad \Phi' = -\Phi, \quad \Psi' = -\Psi, \\ b) \quad & t' = -t, \quad y' = -y. \end{aligned} \quad (6)$$

The corresponding invariant solutions must have the form

$$\Phi = F(x, y + vt), \quad \Psi = G(x, y + vt). \quad (7)$$

Here the RHS functions F and G are antisymmetric with respect to their first argument and symmetric with respect to their second argument:

$$\begin{aligned} F(-x, y + vt) &= -F(x, y + vt), & F(x, -(y + vt)) &= F(x, y + vt), \\ G(-x, y + vt) &= -G(x, y + vt), & G(x, -(y + vt)) &= G(x, y + vt). \end{aligned} \quad (8)$$

In contrast with the KdV model, the value of the constant v in (7) is essential, since no similarity transform connecting solutions with different v 's exists for the Hasegawa–Mima model.

Now let us try to obtain the single solution (7) with some definite value of v . Inserting the form (7) in (1) and taking into account the homogeneous boundary condition

$$\Phi = 0, \quad \Psi = 0 \quad \text{as} \quad |r| \rightarrow \infty \quad (9)$$

we readily find that only trivial zero smooth solution of this form exists.

Thus, the most symmetric non-trivial smooth localized solutions for the Hasegawa–Mima model must contain the finite or infinite sum of terms with different velocities v_1, v_2, \dots :

$$\begin{aligned} \Phi &= F_1(x, y + v_1 t) + F_2(x, y + v_2 t) + \dots, \\ \Psi &= G_1(x, y + v_1 t) + G_2(x, y + v_2 t) + \dots, \end{aligned} \quad (10)$$

where F_i, G_i are antisymmetric functions of their first argument and symmetric functions of their second argument.

Now let us consider the most symmetric periodic in x, y solutions of the Hasegawa–Mima equations. The simplest solutions of this kind are as follows:

$$\Phi = \alpha \sin(k_1 x) \quad \text{and} \quad \Phi = \beta \sin(k_1 x) \cos(\omega t + k_2 y), \quad (11)$$

where ω satisfies the dispersion relation (3). The first solution represents the shear flow along the Oy axis, the second one is the standing wave. Both solutions are trivial inasmuch as the nonlinear term vanishes.

The most symmetric non-trivial periodic solutions for the Hasegawa–Mima equations must have the form (summation over some integer values of m, n , and l is assumed):

$$\Phi = \sum \Phi_{mnl} \sin(mk_1 x) \cos(nk_2 y + \omega_{mnl} t). \quad (12)$$

Here Φ_{mnl} and ω_{mnl} are the functions of k_1 and k_2 to be determined from the equations (1). In this way periodicity and symmetry property (8) will be guaranteed. Analytical solutions of this form (except the trivial ones, like (11)) are not known.

Choosing the initial conditions compatible with the symmetry (8), we can proceed in two ways: to find numerical solutions of the Hasegawa–Mima equations or to build perturbative solutions treating the amplitudes as small but finite parameters. For example, numerical simulations were performed and perturbative solution was obtained in [2] for the combination of the shear flow and the standing wave (11). These solutions which keep the symmetric form (12) are characterized by higher harmonics generation and frequency shifts.

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