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# $n$th Discrete KP Hierarchy

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We report an infinite class of discrete hierarchies which naturally generalize familiar discrete KP one.

## 1 Introduction

The interrelation between discrete and differential integrable hierarchies plays crucial role in obtaining solutions to the discrete multi-matrix models [1, 2]. At a level of KP-type differential hierarchies the discrete structure of multi-matrix models is captured by the Darboux–Bäcklund (DB) transformations. In turn partition functions of multi-matrix models turns out to be $\tau$-functions of differential hierarchies and are constructed as DB orbits of certain simple initial conditions [2]. The well known discrete KP (1-Toda lattice) hierarchy [3] together with its reductions can be viewed as a container for a set of KP-type differential hierarchies whose solutions are generated by DB transformations.

This paper is designed to exhibit certain class of discrete hierarchies which generalize discrete KP and show the relationship with general (unconstrained) differential KP. This relationship yields bi-infinite sequences of differential KP equipped with two compatible gauge transformations. We believe that these results might be of potential interest from the physical point of view.

## 2 $n$th discrete KP

Given the shift operator $\Lambda = (\delta_{i,j-1})_{i,j\in\mathbb{Z}}$ one considers the Lie algebra of pseudo-difference operators

$$\mathcal{D} = \left\{ \sum_{-\infty<k\ll\infty} \ell_k \Lambda^k \right\} = \mathcal{D}_- + \mathcal{D}_+$$

with usual splitting into "negative" and "positive" parts:

$$\mathcal{D}_- = \left\{ \sum_{-\infty<k\leq -1} \ell_k \Lambda^k \right\} \qquad \text{and} \qquad \mathcal{D}_+ = \left\{ \sum_{0<k\ll\infty} \ell_k \Lambda^k \right\}.$$

We assume that entries of bi-infinite diagonal matrices $\ell_k \equiv (\ell_k(i))_{i\in\mathbb{Z}}$ may depend on "spectral" parameter $z$ and multi-time $t \equiv (t_1 \equiv x, t_2, t_3, \ldots)$. In what follows $\partial \equiv \partial/\partial x$ and $\partial_p \equiv \partial/\partial t_p$.

Let us define[1]

$$Q = \Lambda + a_0 z^{n-1}\Lambda^{1-n} + a_1 z^{2(n-1)}\Lambda^{1-2n} + \cdots \in \mathcal{D}, \qquad n \in \mathbb{N} \tag{1}$$

with $a_k = (a_k(i))_{i\in\mathbb{Z}}$ being functions on $t$ only.

[1]Here $z$ acts as component-wise multiplication.

**Proposition 1.** *Lax equations of $Q$-deformations*

$$z^{p(n-1)}\partial_p Q = \left[Q_+^{pn}, Q\right], \qquad p = 1, 2, \ldots \tag{2}$$

*make sense.*

**Proof.** One needs to use standard simple arguments to prove correctness of equations (2). It is enough to show that $\left[Q_+^{pn}, Q\right] = -\left[Q_-^{pn}, Q\right]$ is of the same form as l.h.s. of (2). ■

We will refer to (2) as $n$th discrete KP hierarchy. Let us represent $Q$ as a dressing up of $\Lambda$ by a "wave" operator as $Q = W\Lambda W^{-1}$ where

$$W = I + w_1 z^{n-1}\Lambda^{-n} + w_2 z^{2(n-1)}\Lambda^{-2n} + w_3 z^{3(n-1)}\Lambda^{-3n} + \cdots \in I + \mathcal{D}_-.$$

Then $Q$-deformations are induced by $W$-deformations

$$z^{p(n-1)}\partial_p W = Q_+^{pn} W - W\Lambda^{pn},$$
$$z^{p(n-1)}\partial_p \left(W^{-1}\right)^T = \left(W^{-1}\right)^T \Lambda^{-pn} - \left(Q_+^{pn}\right)^T \left(W^{-1}\right)^T. \tag{3}$$

Define $\chi(t,z) = \left(z^i e^{\xi(t,z)}\right)_{i\in\mathbb{Z}}$, $\chi^*(t,z) = \left(z^{-i} e^{-\xi(t,z)}\right)_{i\in\mathbb{Z}}$ with $\xi(t,z) \equiv \sum_{p=1}^{\infty} t_p z^p$ and wave vectors

$$\Psi(t,z) = W\chi(t,z), \qquad \Psi^*(t,z) = \left(W^{-1}\right)^T \chi^*(t,z). \tag{4}$$

Discrete linear system

$$Q\Psi(t,z) = z\Psi(t,z), \qquad Q^T\Psi^*(t,z) = z\Psi^*(t,z),$$
$$z^{p(n-1)}\partial_p \Psi = Q_+^{pn}\Psi, \qquad z^{p(n-1)}\partial_p \Psi^* = -(Q_+^{pn})^T\Psi^* \tag{5}$$

are evident consequence of (3) and (4). Making use of obvious relations $z\chi = \Lambda\chi$ and $\chi_i = \partial^{i-j}\chi_j$ with $i$ and $j$ being arbitrary integers, we deduce

$$\Psi_i(t,z) = z^i\left(1 + w_1(i)z^{-1} + w_2(i)z^{-2} + \cdots\right)e^{\xi(t,z)}$$
$$= z^i\left(1 + w_1(i)\partial^{-1} + w_2(i)\partial^{-2} + \cdots\right)e^{\xi(t,z)} \equiv z^i\hat{w}_i(\partial)e^{\xi(t,z)} \equiv z^i\psi_i(t,z).$$

What we are going to do next is to establish equivalence of $n$th discrete KP to bi-infinite sequence of differential KP copies "glued" together by two compatible gauge transformations one of which can be recognized as DB transformation mapping $\mathcal{Q}_i \equiv \hat{w}_i\partial\hat{w}_i^{-1}$ to $\mathcal{Q}_{i+n} \equiv \hat{w}_{i+n}\partial\hat{w}_{i+n}^{-1}$. By straightforward calculations one can prove

**Proposition 2.** *The following three statements are equivalent*

*i) the wave vector $\Psi(t,z)$ satisfies discrete linear system*

$$Q\Psi(t,z) = z\Psi(t,z), \qquad z^{n-1}\partial\Psi = Q_+^n\Psi; \tag{6}$$

*ii) the components $\psi_i$ of a vector $\psi \equiv (\psi_i = z^{-i}\Psi_i)_{i\in\mathbb{Z}}$ satisfy*

$$G_i\psi_i(t,z) = z\psi_{i+n-1}(t,z), \qquad H_i\psi_i(t,z) = z\psi_{i+n}(t,z) \tag{7}$$

*with $H_i \equiv \partial - \sum_{s=1}^{n} a_0(i+s-1)$ and*

$$G_i \equiv H_i + a_0(i+n-1) + a_1(i+n-1)H_{i-n}^{-1} + a_2(i+n-1)H_{i-2n}^{-1}H_{i-n}^{-1} + \cdots;$$

*iii) for sequence of dressing operators $\hat{w}_i$ following equations*

$$G_i\hat{w}_i = \hat{w}_{i+n-1}\partial, \qquad H_i\hat{w}_i = \hat{w}_{i+n}\partial \tag{8}$$

*hold.*

Consistency condition of (6) is given by Lax equation

$$z^{n-1}\partial Q = \left[Q_+^n, Q\right] \tag{9}$$

which in explicit form looks as

$$\begin{aligned}\partial a_k(i) &= a_{k+1}(i+n) - a_{k+1}(i)\\ &\quad + a_k(i)\left(\sum_{s=1}^{n} a_0(i+s-1) - \sum_{s=1}^{n} a_0(i+s-(k+1)n)\right), \qquad k \geq 0.\end{aligned} \tag{10}$$

**Remark 1.** One-field reductions of the systems (10) lead to Bogoyavlenskii lattices [4]

$$\partial r_i = r_i\left(\sum_{s=1}^{n-1} r_{i+s} - \sum_{s=1}^{n-1} r_{i-s}\right), \qquad r_i \equiv a_0(i)$$

including well known Volterra lattice $\partial r_i = r_i(r_{i+1} - r_{i-1})$ in the case $n = 2$.

Consistency condition of (8) is given by relations

$$G_{i+n}H_i = H_{i+n-1}G_i, \qquad i \in \mathbb{Z} \tag{11}$$

which in fact are equivalent to (9).

**Proposition 3.** *By virtue of (8) and its consistency condition, Lax operators $\mathcal{Q}_i$ are connected with each other by two invertible compatible gauge transformations*

$$\mathcal{Q}_{i+n-1} = G_i\mathcal{Q}_iG_i^{-1}, \qquad \mathcal{Q}_{i+n} = H_i\mathcal{Q}_iH_i^{-1}. \tag{12}$$

**Proof.** By virtue of (8), we have

$$\mathcal{Q}_{i+n-1} = \hat{w}_{i+n-1}\partial\hat{w}_{i+n-1}^{-1} = \left(G_i\hat{w}_i\partial^{-1}\right)\partial\left(\partial\hat{w}_i^{-1}G_i^{-1}\right) = G_i\hat{w}_i\partial\hat{w}_i^{-1}G_i^{-1} = G_i\mathcal{Q}_iG_i^{-1}.$$

The similar arguments are applied to show second relation in (12). The mapping $\mathcal{Q}_i \to \tilde{\mathcal{Q}}_i = \mathcal{Q}_{i+n-1}$ we denote as $s_1$, while $s_2$ stands for transformation $\mathcal{Q}_i \to \overline{\mathcal{Q}}_i = \mathcal{Q}_{i+n}$. As for compatibility of $s_1$ and $s_2$, by virtue of (11), we have

$$\begin{aligned}\mathcal{Q}_{i+2n-1} &= G_{i+n}\mathcal{Q}_{i+n}G_{i+n}^{-1} = G_{i+n}H_i\mathcal{Q}_iH_i^{-1}G_{i+n}^{-1}\\ &= H_{i+n-1}G_i\mathcal{Q}_iG_i^{-1}H_{i+n-1}^{-1} = H_{i+n-1}\mathcal{Q}_{i+n-1}H_{i+n-1}^{-1}.\end{aligned}$$

So we can write $s_1 \circ s_2 = s_2 \circ s_1$. The inverse maps $s_1^{-1}$ and $s_2^{-1}$ are well defined by the formulas $\mathcal{Q}_{i-n+1} = G_{i-n+1}^{-1}\mathcal{Q}_iG_{i-n+1}$ and $\mathcal{Q}_{i-n} = H_{i-n}^{-1}\mathcal{Q}_iH_{i-n}$. ■

It is obvious that relation $s_1^n = s_2^{n-1}$ holds. Indeed the l.h.s. and r.h.s. of this relation correspond to the same mapping $\mathcal{Q}_i \to \mathcal{Q}_{i+n(n-1)}$. The Abelian group generated by $s_1$ and $s_2$ we denote by symbol $\mathcal{G}$.

Rewrite second equation in (7) as $z^{n-1}H_i\Psi_i(t,z) = \Psi_{i+n}(t,z) = (\Lambda^n\Psi)_i$. From this we derive

$$z^{k(1-n)}(\Lambda^{kn}\Psi)_i = H_{i+(k-1)n}\cdots H_{i+n}H_i\Psi_i,$$
$$z^{k(n-1)}(\Lambda^{-kn}\Psi)_i = H_{i-kn}^{-1}\cdots H_{i-2n}^{-1}H_{i-n}^{-1}\Psi_i.$$

These relations make connection between matrices of the form $P = \sum_{k\in\mathbb{Z}} z^{k(1-n)}p_k(t)\Lambda^{kn}$ and sequences of pseudo-differential operators $\{\mathcal{P}_i,\ i \in \mathbb{Z}\}$ mapping the upper triangular part of given matrix (including main diagonal) into the differential parts of $\mathcal{P}_i$'s and the lower triangular part of the matrix to the purely pseudo-differential parts. More exactly, we have $(P\Psi)_i = \mathcal{P}_i\Psi_i$, $(P_-\Psi)_i = (\mathcal{P}_i)_-\Psi_i$ and $(P_+\Psi)_i = (\mathcal{P}_i)_+\Psi_i$, where

$$\mathcal{P}_i = \sum_{k>0} p_{-k}(i,t)H_{i-kn}^{-1}\cdots H_{i-2n}^{-1}H_{i-n}^{-1} + \sum_{k\geq 0} p_k(i,t)H_{i+(k-1)n}\cdots H_{i+n}H_i = (\mathcal{P}_i)_- + (\mathcal{P}_i)_+.$$

**Proposition 4.** *Equations* $z^{p(n-1)}\partial_p\Psi = Q_+^{pn}\Psi$, $p = 2, 3, \ldots$ *lead to* $\partial_p\psi_i = (\mathcal{Q}_i^p)_+\psi_i$, $p = 2, 3, \ldots$.

**Proof.** We have

$$z^{p(1-n)}(Q^{pn}\Psi)_i = z^p\Psi_i = z^{i+p}\hat{w}_i e^{\xi(t,z)} = z^i\hat{w}_i\partial^p e^{\xi(t,z)} = z^i\hat{w}_i\partial^p\hat{w}_i^{-1}\psi_i = z^i\mathcal{Q}_i^p\psi_i = \mathcal{Q}_i^p\Psi_i.$$

Thus

$$z^{p(n-1)}\partial_p\Psi_i = z^{i+p(n-1)}\partial_p\psi_i = (Q_+^{pn}\Psi)_i = z^{p(n-1)}(\mathcal{Q}_i^p)_+\Psi_i = z^{i+p(n-1)}(\mathcal{Q}_i^p)_+\psi_i.$$

The latter proves proposition. ■

Let us establish equations managing $G_i$- and $H_i$-evolutions with respect to KP flows. Differentiating l.h.s. and r.h.s. of (8) by virtue of Sato–Wilson equations $\partial_p\hat{w}_i = (\mathcal{Q}_i^p)_+\hat{w}_i - \hat{w}_i\partial^p$ formally leads to evolution equations

$$\begin{aligned}
&\partial_p G_i = \left(\mathcal{Q}_{i+n-1}^p\right)_+ G_i - G_i\left(\mathcal{Q}_i^p\right)_+,\\
&\partial_p H_i = \left(\mathcal{Q}_{i+n}^p\right)_+ H_i - H_i\left(\mathcal{Q}_i^p\right)_+.
\end{aligned} \tag{13}$$

Standard arguments can be used to show that equations (13) are properly defined individually. Let us show that permutation relations (11) are invariant under the flows given by equations (13). We have

$$\begin{aligned}
\partial_p(H_{i+n-1}G_i) &= \left\{\left(\mathcal{Q}_{i+2n-1}^p\right)_+ H_{i+n-1} - H_{i+n-1}\left(\mathcal{Q}_{i+n-1}^p\right)_+\right\} G_i\\
&+ H_{i+n-1}\left\{\left(\mathcal{Q}_{i+n-1}^p\right)_+ G_i - G_i\left(\mathcal{Q}_i^p\right)_+\right\} = \left(\mathcal{Q}_{i+2n-1}^p\right)_+ H_{i+n-1}G_i - H_{i+n-1}G_i\left(\mathcal{Q}_i^p\right)_+\\
&= \left(\mathcal{Q}_{i+2n-1}^p\right)_+ G_{i+n}H_i - G_{i+n}H_i\left(\mathcal{Q}_i^p\right)_+ = \left\{\left(\mathcal{Q}_{i+2n-1}^p\right)_+ G_{i+n} - G_{i+n}\left(\mathcal{Q}_{i+n}^p\right)_+\right\} H_i\\
&+ G_{i+n}\left\{\left(\mathcal{Q}_{i+n}^p\right)_+ H_i - H_i\left(\mathcal{Q}_i^p\right)_+\right\} = \partial_p(G_{i+n}H_i).
\end{aligned}$$

Hence we proved that evolution equations (13) are consistent.

Define $\Phi_i = \Phi_i(t)$ via $H_i\Phi_i = 0$ or equivalently through equation $\partial\Phi_i = \Phi_i\sum\limits_{s=1}^{n} a_0(i+s-1)$. Taking into consideration second equation in (13), we have

$$\partial_p(H_i\Phi_i) = \left(\mathcal{Q}_{i+n}^p\right)_+ H_i\Phi_i - H_i\left(\mathcal{Q}_i^p\right)_+\Phi_i + H_i\partial_p\Phi_i = 0.$$

From this we derive $\partial_p\Phi_i = \left(\mathcal{Q}_i^p\right)_+\Phi_i + \alpha_i\Phi_i$ where $\alpha_i$'s are some constants. Commutativity condition $\partial_p\partial_q\Phi_i = \partial_q\partial_p\Phi_i$ leads to evolution equations for KP eigenfunctions $\partial_p\Phi_i = \left(\mathcal{Q}_i^p\right)_+\Phi_i$, i.e. $\alpha_i = 0$. Thus the relations $\mathcal{Q}_{i+n} = H_i\mathcal{Q}_iH_i^{-1}$ defines DB transformations with eigenfunctions $\Phi_i = \tau_{i+n}/\tau_i$. It should perhaps to recall that arbitrary eigenfunction of Lax operator $\mathcal{Q}$ contains information about DB transformation $\tau \to \overline{\tau} = \Phi\tau$ while the identity[2]

$$\left\{\tau\left(t - \left[z^{-1}\right]\right), \overline{\tau}(t)\right\} + z\left(\tau\left(t - \left[z^{-1}\right]\right)\overline{\tau}(t) - \overline{\tau}\left(t - \left[z^{-1}\right]\right)\tau(t)\right) = 0$$

holds.

So, we have shown that $n$th discrete KP is equivalent to sequence of differential KP linked with each other by two compatible gauge transformations one of which, namely, $s_2 : \mathcal{Q}_i \to \mathcal{Q}_{i+n}$ are nothing but Darboux–Bäcklund transformation. The problem which can be addressed is to describe $n$th discrete KP in the language of bilinear identities by analogy as was done for ordinary discrete KP [5].

[2]Here conventional notations $\{f, g\} = \partial f \cdot g - \partial g \cdot f$ and $\left[z^{-1}\right] = \left(1/z, 1\left(2z^2\right), \ldots\right)$ are used.

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