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Integrable Hamiltonian Systems via Quasigraded Lie Algebras

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In the present paper we construct integrable Hamiltonian systems of the Euler–Arnold type associated with infinite-dimensional quasigraded Lie algebras of matrix valued functions on higher genus curves. In details is considered the case when underlying matrix Lie algebra coincide with $gl(n)$. Corresponding generalizations of Steklov integrable systems as long as $gl(n)$ analogues of Clebsh integrable systems are obtained.

1 Introduction

The main purpose of the present paper is to construct new integrable Hamiltonian systems of the Euler–Arnold type. Our approach to the solution of this problem is based on the usage of infinite-dimensional Lie algebras. Traditionally Lie theoretical explanation of the integrability of Euler–Arnold equations on finite-dimensional Lie algebras is based on on the loop algebras and Kostant–Adler scheme [1, 2]. In the papers [3, 4] it was shown, that in similar way integrable Euler–Arnold equations on the algebra $so(3)$ and some its extensions could be obtained from the infinite-dimensional Lie algebras of the special elliptic functions with the values in $so(3)$. In our previous papers [5, 6] we generalized construction described in [4] for the case of classical matrix algebras of higher ranks. Growth of the rank of algebra requires automatic growth of the genus of the curve. In the result we have obtained algebras of $gl(n)$-, $so(n)$- and $sp(n)$-valued functions on the algebraic curves of higher genus. The most important property of the discovered algebras is that they admit Kostant–Adler scheme, and hence, could be used to construct new integrable systems. Using them we have constructed new integrable Hamiltonian systems on the Lie algebras $so(n) \oplus so(n)$, $so(n) + so(n)$, $e(n)$ that generalize integrable systems of Steklov–Veselov, Steklov–Liapunov, and Clebsh [5, 6, 7].

In the present paper we consider the case when underlying matrix Lie algebra coincides with $\mathfrak{g} = gl(n)$. We show that there exist precise integrable $gl(n)$-analogues of Steklov–Veselov and Steklov–Liapunov systems on $gl(n) \oplus gl(n)$, $gl(n) + gl(n)$ along with $gl(n)$ analogue of the Clebsh system on $gl(n-1) + \mathbb{R}^{2n}$. It is necessary to notice that same results are valid for the case of $\mathfrak{g} = sp(n)$. We do not adduce them here due to the restricted size of the article.

2 Quasi-graded algebras on higher genus curves

2.1 Construction

1. Higher genus curve embedded in $\mathbb{C}^n$*.* Let us consider in the space $\mathbb{C}^n$ with the coordinates $w_1, w_2, \ldots, w_n$ the following system of quadrics:

$$w_i^2 - w_j^2 = a_j - a_i, \qquad i, j = \overline{1, n}, \tag{1}$$

where a_i are arbitrary complex numbers. Rank of this system is $n-1$, so substitution:

$$w_i^2 = w - a_i, \qquad y = \prod_{i=1}^{n} w_i, \qquad y^2 = \prod_{i=1}^{n} w_i^2$$

solves these equations and defines the equation of the hyperelliptic curve $\mathcal{H}$.

2. Classical Lie algebras. Let $\mathfrak{g}$ denotes one of the classical matrix Lie algebras $gl(n)$, $so(n)$ and $sp(n)$ over the field of the complex numbers. We will need explicit form of their bases. Let $I_{i,j} \in \mathrm{Mat}\,(n, C)$ be a matrix defined as:

$$(I_{ij})_{ab} = \delta_{ia}\delta_{jb}.$$

Evidently, a basis in the algebra $gl(n)$ could be built from the matrices $X_{ij} \equiv I_{ij}$, $i, j \in 1, \ldots, n$. The commutation relations in $gl(n)$ will have the standard form:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j} X_{i,l} - \delta_{i,l} X_{k,j}.$$

The basis in the algebra $so(n)$ could be chosen as: $X_{ij} \equiv I_{ij} - I_{i,j}$, $i, j \in 1, \ldots, n$, with "skew-symmetry" property $X_{ij} = -X_{ji}$ and the following commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j} X_{i,l} - \delta_{i,l} X_{k,j} + \delta_{j,l} X_{k,i} - \delta_{k,i} X_{j,l}.$$

The basis in the algebra $sp(n)$ we choose as $X_{ij} = I_{ij} - \epsilon_i \epsilon_j I_{-i,-j}$, $|i|, |j| \in 1, \ldots, n$, with the property $X_{i,j} = -\epsilon_i \epsilon_j X_{-j,-i}$, where $\epsilon_j = \operatorname{sign} j$ and commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j} X_{i,l} - \delta_{i,l} X_{k,j} + \epsilon_i \epsilon_j (\delta_{j,-l} X_{k,-i} - \delta_{k,-i} X_{-j,l}).$$

3. Algebras on the curve. For the basic elements X_{ij} of all three algebras $gl(n)$, $so(n)$ and $sp(n)$ and arbitrary $n \in \mathbb{Z}$ we introduce the following algebra-valued functions on the curve $\mathcal{H}$, or to be more precise on its ramified covering:

$$X_{ij}^n = X_{ij} \otimes w^n w_i w_j.$$

The next theorem holds true:

Theorem 1. *(i) Elements X_{ij}^n form $n \in \mathbb{Z}$ quasi-graded Lie algebra $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ with the following commutation relations:*

$$1)\ [X_{ij}^n, X_{kl}^m] = \delta_{kj} X_{il}^{n+m+1} - \delta_{il} X_{kj}^{n+m+1} + a_i \delta_{il} X_{kj}^{n+m} - a_j \delta_{kj} X_{il}^{n+m} \quad \text{for the } gl(n); \tag{2a}$$

$$\begin{aligned} 2)\ [X_{ij}^n, X_{kl}^m] &= \delta_{kj} X_{il}^{n+m+1} - \delta_{il} X_{kj}^{n+m+1} + \delta_{jl} X_{ki}^{n+m+1} - \delta_{ik} X_{jl}^{n+m+1} \\ &+ a_i \delta_{il} X_{kj}^{n+m} - a_j \delta_{kj} X_{il}^{n+m} + a_i \delta_{ik} X_{jl}^{n+m} - a_j \delta_{jl} X_{ki}^{n+m} \quad \text{for the } so(n); \end{aligned} \tag{2b}$$

$$\begin{aligned} 3)\ [X_{ij}^n, X_{kl}^m] &= \delta_{kj} X_{il}^{n+m+1} - \delta_{il} X_{kj}^{n+m+1} + \epsilon_i \epsilon_j \left(\delta_{j-l} X_{k-i}^{n+m+1} - \delta_{i-k} X_{j-l}^{n+m+1} \right) \\ &+ a_i \delta_{il} X_{kj}^{n+m} - a_j \delta_{kj} X_{il}^{n+m} + a_i \epsilon_i \epsilon_j \left(a_i \delta_{i-k} X_{j-l}^{n+m} - a_j \delta_{j-l} X_{k-i}^{n+m} \right) \quad \text{for the } sp(n). \end{aligned} \tag{2c}$$

(ii) Algebra $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ as a linear space admits a decomposition into the direct sum of two subalgebras: $\widetilde{\mathfrak{g}}_{\mathcal{H}} = \widetilde{\mathfrak{g}}_{\mathcal{H}}^+ + \widetilde{\mathfrak{g}}_{\mathcal{H}}^-$, where subalgebras $\widetilde{\mathfrak{g}}_{\mathcal{H}}^+$ and $\widetilde{\mathfrak{g}}_{\mathcal{H}}^-$ are generated by the elements X_{ij}^0, and X_{ij}^{-1} correspondingly.

Example 1. Let $\mathfrak{g} = so(3)$. In this case putting $X_k \equiv \epsilon_{ijk} X_{ij}$, we obtain the following commutation relations:

$$[X_i^n, X_j^m] = \epsilon_{ijk} X_k^{n+m+1} + \epsilon_{ijk} a_k X_k^{n+m}.$$

Remark 1. From the item (i) of the theorem it follows that in the rational degeneration, i.e. when $a_i = 0$, $\widetilde{\mathfrak{g}}_{\mathcal{H}} = \widetilde{\mathfrak{g}}$, where $\widetilde{\mathfrak{g}}$ is an ordinary loop algebra.

2.2 Coadjoint representation

To define the coadjoint representation we have to define $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$. For our purposes it will be convenient to identify $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ with $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ as linear spaces. In order to do this we will define pairing between $L(w) \in \widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ and $X(w) \in \widetilde{\mathfrak{g}}_{\mathcal{H}}$ in the following way:

$$\langle X(w), L(w)\rangle_f = c_n \operatorname*{res}_{w=0} y^{-2}(w)(X(w)|L(w)), \tag{3}$$

where $f(w)$ is arbitrary function on the curve $\mathcal{H}$. It is easy to show that element dual to X_{ij}^{-m} with respect to this pairing is $Y_{ij}^m \equiv (X_{ij}^{-m})^* = \frac{w^{m-1}y^2(w)}{w_iw_j}X^*_{ij}$. Hence the general element of the dual space has the following form:

$$L(w) = \sum_{m\in Z}\sum_{i,j=1}^{n} l_{ij}^{(m)}\frac{w^{m-1}y^2(w)}{w_iw_j}X^*_{ij}. \tag{4}$$

Coadjoint action of algebra $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ on its dual space $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ coincides with commutator:

$$\mathrm{ad}^*_{X(w)}L(w) = [L(w), X(w)]. \tag{5}$$

From the explicit form of coadjoint action (5) follows the next statement:

Proposition 1. *Functions* $I_m^k(L(w)) = \operatorname*{res}_{w=0} w^{-m-1}\operatorname{Tr}L(w)^k$*, where* $m\in\mathbb{Z}$*, are invariants of coadjoint representation.*

3 Integrable systems from hyperelliptic algebras

3.1 Poisson structures and Poisson subspaces

1. First Lie–Poisson structure. In the space $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ we can define Lie–Poisson brackets using introduced above pairing (3). It defines brackets on $P(\widetilde{\mathfrak{g}}^*_{\mathcal{H}})$ in the following way:

$$\{F(L), G(L)\} = \sum_{l,m\in Z}\sum_{i,j,p,s=1}^{n}\langle L(w), [X_{ij}^{-l}, X_{ps}^{-m}]\rangle\frac{\partial G}{\partial l_{ij}^{(l)}}\frac{\partial F}{\partial l_{ps}^{(m)}}. \tag{6}$$

From the Proposition 1 follows the next statement:

Proposition 2. *Functions* $I_m^k(L(w))$ *are central for brackets* $\{\ ,\ \}$.

Let us explicitly calculate Poisson brackets (6). Taking into account that $l_{ij}^{(m)} = \langle L(w), X_{ij}^{-m}\rangle$, it is easy to show, that for the coordinate functions $l_{ij}^{(m)}$ these brackets have the following form:

$$1)\ \left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\} = \delta_{kj}l_{il}^{(n+m-1)} - \delta_{il}l_{kj}^{(n+m-1)} + a_i\delta_{il}l_{kj}^{(n+m)} - a_j\delta_{kj}l_{il}^{(n+m)} \quad \text{for the } gl(n); \tag{7a}$$

$$\begin{aligned}2)\ \left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\} &= \delta_{kj}l_{il}^{(n+m-1)} - \delta_{il}l_{kj}^{(n+m-1)} + \delta_{jl}l_{ki}^{(n+m-1)} - \delta_{ik}l_{jl}^{(n+m-1)}\\ &+ a_i\delta_{il}l_{kj}^{(n+m)} - a_j\delta_{kj}l_{il}^{(n+m)} + a_i\delta_{ik}l_{jl}^{(n+m)} - a_j\delta_{jl}l_{ki}^{(n+m)} \quad \text{for the } so(n);\end{aligned} \tag{7b}$$

$$\begin{aligned}3)\ \left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\} &= \delta_{kj}l_{il}^{(n+m-1)} - \delta_{il}l_{kj}^{(n+m-1)} + \epsilon_i\epsilon_j\left(\delta_{j-l}l_{k-i}^{(n+m-1)} - \delta_{i-k}l_{j-l}^{(n+m-1)}\right)\\ &+ a_i\delta_{il}l_{kj}^{(n+m)} - a_j\delta_{kj}l_{il}^{(n+m)} + \epsilon_i\epsilon_j\left(a_i\delta_{i-k}l_{j-l}^{(n+m)} - a_j\delta_{j-l}l_{k-i}^{(n+m)}\right) \quad \text{for the } sp(n).\end{aligned} \tag{7c}$$

2. Second Lie–Poisson structure. Let us introduce into the space $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ new Poisson brackets $\{\ ,\ \}_0$, which are a Lie–Poisson brackets for the algebra $\widetilde{\mathfrak{g}}^0_{\mathcal{H}}$, where $\widetilde{\mathfrak{g}}^0_{\mathcal{H}} = \widetilde{\mathfrak{g}}^-_{\mathcal{H}} \ominus \widetilde{\mathfrak{g}}^+_{\mathcal{H}}$. Explicitly, this brackets have the following form:

$$\{l^{(n)}_{ij}, l^{(m)}_{kl}\}_0 = -\{l^{(n)}_{ij}, l^{(m)}_{kl}\}, \quad n, m \in \mathbb{Z}_+, \qquad \{l^{(n)}_{ij}, l^{(m)}_{kl}\}_0 = \{l^{(n)}_{ij}, l^{(m)}_{kl}\}, \quad n, m \in \mathbb{Z}_- \cup 0,$$
$$\{l^{(n)}_{ij}, l^{(m)}_{kl}\}_0 = 0, \quad m \in \mathbb{Z}_- \cup 0, \quad n \in \mathbb{Z}_+ \quad \text{or} \quad n \in \mathbb{Z}_- \cup 0, \quad m \in \mathbb{Z}_+.$$

Let subspace $\mathcal{M}_{s,p} \subset \widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ be defined as follows:

$$\mathcal{M}_{s,p} = \sum_{m=-s+1}^{p} (\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_m.$$

Brackets $\{\ ,\ \}_0$ could be correctly restricted to $\mathcal{M}_{s,p}$. It follows from the next proposition:

Proposition 3. *Subspaces* $\mathcal{J}_{p,s} = \sum\limits_{m=-\infty}^{-p-1} (\widetilde{\mathfrak{g}}_{\mathcal{H}})_m + \sum\limits_{m=s}^{\infty} (\widetilde{\mathfrak{g}}_{\mathcal{H}})_m$ *are ideals in* $\widetilde{\mathfrak{g}}^0_{\mathcal{H}}$.

3.2 Algebras of integrals and Hamiltonian equations

To construct integrable Hamiltonian systems we need a large family of mutually commuting functions (integrals of motion). It is provided by the following theorem:

Theorem 2. *Let functions* $\{I^k_m(L)\}$ *be defined as in Proposition 1. Their restriction to* $\mathcal{M}_{s,p}$ *generate commutative algebra with respect to the restriction of the brackets* $\{\ ,\ \}_0$ *on* $\mathcal{M}_{s,p}$.

Dynamical equations we will consider here are Hamiltonian equations of the form:

$$\frac{dl^{(k)}_{ij}}{dt} = \left\{ l^{(k)}_{ij}, H\left(l^{(m)}_{kl}\right)\right\}_0, \tag{8}$$

where Hamiltonian H is one of the functions I^k_m or their linear combination. These equations could be written in the form of Lax type equations [2]:

$$\frac{dL(w)}{dt} = P_{M_{s,p}}([L(w), M(w)]), \tag{9}$$

where $P_{M_{s,p}}$ denotes operator that project dual space onto subspace $M_{s,p}$ $L(w) \in \mathcal{M}_{s,p}$, and second operator is defined as follows: $M(w) = (P_- - P_+)\nabla H(L(w))$. Here $P_\pm$ are projection operators on the subalgebra $\widetilde{\mathfrak{g}}^\pm_{\mathcal{H}}$,

$$\nabla H(L(w)) = \sum_{k=-p}^{s-1} \sum_{ij=1}^{n} \frac{\partial H}{\partial l^{(k)}_{ij}} X^{-k}_{ij} \tag{10}$$

is an algebra-valued gradient of H.

Thus we have constructed Hamiltonian systems possessing (Theorem 2) a lot of mutually commuting integrals of motion. In the next section we will consider examples of such systems.

4 Integrable systems in finite-dimensional quotients

The most interesting from the physical point of view examples usually arise in the spaces $\mathcal{M}_{s,p}$ with small s and p. We will assume, that curve $\mathcal{H}$ is nondegenerated, i.e. $a_i \neq a_j$ for $i \neq j$.

4.1 Generalized $gl(n)$ tops

Let us consider subspace $\mathcal{M}_{0,1}$. It is evident that $\mathcal{M}_{0,1} = (\widetilde{\mathfrak{g}}^{+}_{\mathcal{H}}/\mathcal{J}_{1,0})^* = \mathfrak{g}^*$. Corresponding Lax operator $L(w) \in \mathcal{M}_{0,1}$ has the following form:

$$L(w) = \sum_{i,j=1,k} l^{(1)}_{ij} \frac{y^2(w)}{w_i w_j} X^*_{ij}.$$

Let us consider the case $\mathfrak{g} = gl(n)$. In this case we have: $X^*_{ij} = X_{ji}$. Lie–Poisson brackets between the coordinate functions $l_{ij} \equiv l^{(1)}_{ij}$ have standard form:

$$\{l_{ij}, l_{kl}\} = \delta_{kj} l_{il} - \delta_{il} l_{kj}.$$

Commuting integrals are constructed using expansions in the powers of w of the functions: $I_k(w) = \mathrm{Tr}\,(L(w))^k$. We are especially interested in the quadratic Hamiltonians. Let

$$h(w) \equiv I_2(w) = \sum_{s=0}^{2n-2} h_s(l_{ij}) w^s = \sum_{ij} \left(\prod_{k \neq i,j} (w - a_k)^2 \right) l_{ij} l_{ji}.$$

We obtain:

$$\begin{aligned}
h_0 &= \left(\prod_{k=1}^{n} a_k^2 \right) \sum_{i,j=1}^{n} \frac{l_{ij} l_{ji}}{a_i a_j}, \\
h_1 &= -\left(\prod_{k=1}^{n} a_k^2 \right) \sum_{i,j=1}^{n} \left(2 \sum_{k=1}^{n} a_k^{-1} - \left(a_i^{-1} + a_j^{-1} \right) \right) \frac{l_{ij} l_{ji}}{a_i a_j}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
h_{2n-3} &= -\sum_{i,j=1}^{n} \left(2 \sum_{k=1}^{n} a_k - (a_i + a_j) \right) l_{ij} l_{ji}, \\
h_{2n-2} &= \sum_{i,j=1}^{n} l_{ij} l_{ji}.
\end{aligned}$$

Last function in this set is a Casimir function, previous $2n-3$ define nontrivial flows on each coadjoint orbit in $\mathfrak{g}^*$. For the Hamiltonian of the generalized $gl(n)$ rigid body we can take $H(l_{ij}) \equiv 1/2 h_{n-1}(l_{ij})$ or $H(l_{ij}) \equiv 1/2 h_0(l_{ij})$. They are transformed to the standard Hamiltonian of the Euler top in the case $n = 3$ after reduction to $so(n)$ subalgebra.

4.2 Generalized $gl(n-1)$ Clebsh systems

Let us consider subspace $\mathcal{M}_{1,0}$. Corresponding Lax matrix $L(w) \in \mathcal{M}_{1,0}$ has the following form:

$$L(w) = w^{-1} \sum_{i,j=1,n} l^{(0)}_{ij} \frac{y^2(w)}{w_i w_j} X_{ji}.$$

In the space $\mathcal{M}_{1,0}$ Poisson structure $\{\ ,\ \}$ has the following form:

$$\left\{ l^{(0)}_{ij}, l^{(0)}_{kl} \right\} = a_i \delta_{il} l^{(0)}_{kj} - a_j \delta_{kj} l^{(0)}_{il}. \tag{11}$$

The Lie algebraic structure that is defined by these brackets strongly depends on the constants a_i. Let us consider the case of the simplest "degeneration" $a_n \to 0$, $a_i \neq 0$, where $i < n$. In this case we will have the following commutation relations:

$$\left\{l^{(0)}_{ij}, l^{(0)}_{kl}\right\} = a_i\delta_{il}l^{(0)}_{kj} - a_j\delta_{kj}l^{(0)}_{il}, \qquad \left\{l^{(0)}_{ij}, l^{(0)}_{kn}\right\} = -a_j\delta_{kj}l^{(0)}_{in}, \qquad \left\{l^{(0)}_{ij}, l^{(0)}_{nk}\right\} = a_i\delta_{ik}l^{(0)}_{nj},$$
$$\left\{l^{(0)}_{in}, l^{(0)}_{jn}\right\} = \left\{l^{(0)}_{ni}, l^{(0)}_{nj}\right\} = \left\{l^{(0)}_{ij}, l^{(0)}_{nn}\right\} = 0, \qquad \left\{l^{(0)}_{in}, l^{(0)}_{nj}\right\} = a_i\delta_{ij}l^{(0)}_{nn},$$

where $i, j, k < n$. Making the following change of the variables:

$$l_{ij} = \frac{l^{(0)}_{ij}}{b_ib_j}, \quad x_k = \frac{l^{(0)}_{kn}}{b_k}, \quad y_k = \frac{l^{(0)}_{nk}}{b_k}, \quad z = l^{(0)}_{nn}, \quad \text{where } \ b_i = a_i^{1/2}, \ \ i,j,k < n \tag{12}$$

we obtain commutation relations for the Lie algebra $gl(n-1) + H^{2n+1}$:

$$\{l_{ij}, l_{kl}\} = \delta_{il}l_{kj} - \delta_{kj}l_{il}, \qquad \{x_i, y_j\} = z, \qquad \{l_{ij}, x_k\} = -\delta_{kj}x_i,$$
$$\{l_{ij}, y_k\} = \delta_{ik}y_k, \qquad \{x_i, x_j\} = \{y_i, y_j\} = \{l_{ij}, z\} = 0,$$

where H^{2n+1} is a Heisenberg algebra in the space $\mathbb{R}^{2n+1}$. It is evident, that z is a central element in this algebra, so we can put $z = 0$. Corresponding Poisson algebra will coincide with semi-direct sum $gl(n-1) + \mathbb{R}^{2n}$. We will call corresponding integrable Hamiltonian system "$gl(n)$ *Clebsh system*".

Let us calculate commuting integrals of the $gl(n)$ Clebsh system. They are constructed using expansions in the powers of w of the functions: $H_k(w) = \mathrm{Tr}(L(w))^k$. Let us calculate explicitly second order integrals:

$$h(w) \equiv H_2(w) = \sum_{s=-2}^{2n-4} h_s\left(l^{(0)}_{ij}\right) w^s = w^{-2}\sum_{i,j=1,n}\left(\prod_{k\neq i,j}(w-a_k)^2\right) l^{(0)}_{ij}l^{(0)}_{ji}.$$

It is not difficult to notice that in the case $a_n \neq 0$, Hamiltonians have essentially the same form as in the previous example of the generalized tops (modulo the shift the indices $h_k \to h_{k-2}$ and replacing of variables: $l_{ij} \to l^{(0)}_{ij}$). Let us now calculate these Hamiltonians in the limit $a_n \to 0$, $z \to 0$. Taking into account coordinate transformation (12) we obtain:

$$h_{-2} = 2\left(\prod_{k=1}^{n-1} a_k^2\right)\sum_{k=1}^{n-1} x_ky_k,$$
$$h_{-1} = (-1)\left(\prod_{k=1}^{n-1} a_k^2\right)\left(\sum_{i,j=1}^{n-1}\left(l_{ij}l_{ji} - 2a_i^{-1}x_iy_i\right) - h_0\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$h_{2n-5} = (-1)\left(\sum_{i,j=1}^{n-1}(a_i+a_j)a_ia_jl_{ij}l_{ji} + 2a_i^2x_iy_i\right) - 2\left(\sum_{k=1}^{n-1}a_k\right)h_{2n-4},$$
$$h_{2n-4} = \left(\sum_{i,j=1}^{n-1} a_ia_jl_{ij}l_{ji} + 2a_ix_iy_i\right).$$

Function h_{-2} is a Casimir function. For the Hamiltonian of the Clebsh system one can take, for example, h_{-1} or h_{2n-4}. They are transformed to the standard integrals of the Clebsh system in the case $n = 3$ after reduction to $so(n)$ subalgebra.

4.3 Generalized interacting $gl(n)$ tops

Let us consider subspace $\mathcal{M}_{1,1}$. In the case $a_i \neq 0$, as it follows from the explicit form of the brackets given below, $\mathcal{M}_{1,1} = (\mathfrak{g} \oplus \mathfrak{g})^*$. Corresponding Lax operator $L(w) \in \mathcal{M}_{1,1}$ has the following form:

$$L(w) = \sum_{i,j=1}^{n} \left(w^{-1} l_{ij}^{(0)} + l_{ij}^{(1)} \right) \frac{y^2(w)}{w_i w_j} X_{ij}^*.$$

In the of $gl(n)$ case we may put $X_{ij}^* = X_{ji}$. Lie–Poisson brackets between the coordinate functions $l_{ij}^{(1)}$ are the following:

$$\left\{ l_{ij}^{(0)}, l_{kl}^{(0)} \right\} = -a_i \delta_{il} l_{kj}^{(0)} + a_j \delta_{kj} l_{il}^{(0)}, \qquad \left\{ l_{ij}^{(1)}, l_{kl}^{(1)} \right\} = \delta_{kj} l_{il}^{(1)} - \delta_{il} l_{kj}^{(1)}, \qquad \left\{ l_{ij}^{(0)}, l_{kl}^{(1)} \right\} = 0.$$

Putting $b_i = a_i^{1/2}$ and making the change of variables: $l_{ij} = l_{ij}^{(1)}$, $m_{ij} = \frac{l_{ij}^{(0)}}{b_i b_j}$, we obtain canonical coordinates of the direct sum of two algebras $gl(n)$:

$$\{m_{i,j}, m_{k,l}\} = \delta_{kj} m_{il} - \delta_{il} m_{kj}, \qquad \{l_{ij}, l_{kl}\} = \delta_{kj} l_{il} - \delta_{il} l_{kj}, \qquad \{l_{ij}, m_{kl}\} = 0.$$

Commuting integrals are constructed using expansion in the powers of w of the functions: $I_k(w) = \operatorname{Tr}(L(w))^k$. We are interested in the quadratic integrals:

$$h(w) \equiv I_2(w) = \sum_{s=-2}^{2n-2} h_s \left(l_{ij}^{(1)} \right) w^s = \sum_{ij} \left(\prod_{k \neq i,j} (w - a_k)^2 \right) \left(l_{ij}^{(0)} + w l_{ij}^{(1)} \right)^2.$$

By direct calculations making the described above change of variables we obtain:

$$h_{-2} = \left(b_1^4 b_2^4 \cdots b_n^4 \right) \sum_{i,j=1}^{n} m_{ij} m_{ji},$$

$$h_{-1} = -\left(b_1^4 b_2^4 \cdots b_n^4 \right) \left(\sum_{i,j=1}^{n} \left(2 \sum_{k=1,n} b_k^{-2} - \left(b_i^{-2} + b_j^{-2} \right) \right) m_{ij} m_{ji} - 2 b_i^{-1} b_j^{-1} m_{ij} l_{ji} \right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$h_{2n-3} = -\left(\sum_{i,j=1}^{n} \left(2 \sum_{k=1}^{n} b_k^2 - \left(b_i^2 + b_j^2 \right) \right) l_{ij} l_{ji} - 2 b_i b_j m_{ij} l_{ji} \right),$$

$$h_{2n-2} = \sum_{i,j=1}^{n} l_{ij} l_{ji}.$$

It is evident that functions h_{-2} and h_{2n-2} are invariants. For the Hamiltonian of the generalized interacting rigid bodies we can take either h_{n-1} or h_1. Operator M and Lax equations for these Hamiltonians are calculated using formulas (9), (10).

4.4 Steklov–Liapunov system on $gl(n) + gl(n)$

Let us consider subspace $\mathcal{M}_{0,2} = (\widetilde{\mathfrak{g}}_{\mathcal{H}}^{+} / \mathcal{J}_{2,0})^*$. It is easy to show that $\mathcal{M}_{0,2} = (\mathfrak{g} + \mathfrak{g})^*$. Corresponding Lax operator $L(w) \in \mathcal{M}_{0,2}$ has the following form:

$$L(w) = \sum_{i,j=1}^{n} \left(l_{ij}^{(1)} + w l_{ij}^{(2)} \right) \frac{y^2(w)}{w_i w_j} X_{ij}^*.$$

We will again be concentrated on $\mathfrak{g} = gl(n)$ case and put $X^*_{ij} = X_{ji}$. Lie–Poisson brackets between coordinate functions are the following:

$$\left\{l^{(1)}_{ij}, l^{(1)}_{kl}\right\} = \delta_{kj} l^{(1)}_{il} - \delta_{il} l^{(1)}_{kj} + a_i \delta_{il} l^{(2)}_{kj} - a_j \delta_{kj} l^{(2)}_{il},$$

$$\left\{l^{(1)}_{ij}, l^{(2)}_{kl}\right\} = \delta_{kj} l^{(2)}_{il} - \delta_{il} l^{(2)}_{kj}, \qquad \left\{l^{(2)}_{ij}, l^{(2)}_{kl}\right\} = 0.$$

Change of variables: $l^{(1)}_{ij} = l_{ij} - a_i p_{ij}$, $l^{(2)}_{ij} = p_{ij}$ transforms described above brackets to the standard brackets on the semi-direct sum $gl(n) + gl(n)$:

$$\{l_{ij}, l_{kl}\} = \delta_{kj} l_{il} - \delta_{il} l_{kj}, \qquad \{l_{ij}, p_{kl}\} = \delta_{kj} p_{il} - \delta_{il} p_{kj}, \qquad \{p_{ij}, p_{kl}\} = 0.$$

Commuting integrals are constructed using expansion in the powers of w of the functions: $I_k(w) = \mathrm{Tr}\,(L(w))^k$. We are again interested mainly in quadratic integrals:

$$h(w) \equiv I_2(w) = \sum_{s=0}^{2n-2} h_{s+2}\left(l^{(1)}_{ij}\right) w^s = w^2 \sum_{ij} \left(\prod_{k\neq i,j} (w - a_k)^2\right) \left(l^{(1)}_{ij} + w l^{(2)}_{ij}\right)^2.$$

By direct calculations, making the described above change of variables we obtain the following set of Hamiltonians:

$$h_0 = \left(a_1^2 a_2^2 \cdots a_n^2\right) \sum_{i,j=1}^{n} \frac{(l_{ij} - a_i p_{ij})(l_{ji} - a_j p_{ji})}{a_i a_j},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$h_{2n-1} = (-1)\left(2\sum_{k=1}^{n} a_k\right) \sum_{i,j=1}^{n} p_{ij} p_{ji} + 2 l_{ij} p_{ji},$$

$$h_{2n} = \sum_{i,j=1}^{n} p_{ij} p_{ji}.$$

Last two functions are invariant functions. If we choose function $H = h_0$ for the Hamiltonian function we obtain precise $gl(n)$ generalization of Steklov–Liapunov system.

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