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On Integrable Quantum System of Particles with Chern–Simons Interaction

Wolodymyr SKRYPNIK

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv-4, Ukraine
E-mail: *skrypnik@imath.kiev.ua*

The Gibbs (grand-canonical) reduced density matrices (RDMs) are calculated in the thermodynamic weak-coupling limit for the non-relativistic spinless system of particles, interacting via collective electromagnetic pair vector Chern–Simons potential, and characterized by the Maxwell–Boltzmann (MB) statistics.

1 Introduction

The Chern–Simons 2-d quantum system of n nonrelativistic spinless identical particles of unit mass is described by the Hamiltonian $\dot{H}_n$, defined on $C^\infty(\mathbb{R}^{2n}\setminus \cup_{j,k}(x_j = x_k))$

$$\dot{H}_n = \frac{1}{2}\sum_{j=1}^{n} ||p_j - a_j(X_n)||^2, \qquad a_j^\nu(X_n) = \epsilon^{\nu\mu}\partial_{\mu,j}U_C(X_n), \quad \nu,\mu = 1,2, \tag{1}$$

$$U_C(X_n) = g\sum_{1\le k<j\le n} e_j e_k \ln|x_j - x_k|, \quad X_n = (x_1,\ldots,x_n)\in\mathbb{R}^{2n}, \quad p_j^\mu = i^{-1}\partial_{\mu,j} = i^{-1}\frac{\partial}{\partial x_j^\mu},$$

where $||v||^2 = \left(v^1\right)^2 + \left(v^2\right)^2$, ϵ is the skew symmetric tensor and the repeating index implies a summation, real number e_j (a charge) takes values in a finite set $E_{\{c\}}$ from $\mathbb{R}$.

Differentiating the equality $f\left(f^{-1}(x)\right) = x$ we derive the formula

$$\frac{df^{-1}(x)}{dx} = \left(\frac{df(y)}{dy}\right)^{-1}, \qquad y = f^{-1}(x).$$

From this equality for $f(x) = \tan x$ and the equality $\frac{d}{dx}\tan x = 1 + \tan^2(x)$ the following relation is derived

$$\frac{\partial}{\partial x^\nu}\arctan\frac{x^2}{x^1} = \epsilon^{\nu\mu}x^\mu|x|^{-2} = \epsilon^{\nu\mu}\frac{\partial}{\partial x^\mu}\ln|x|.$$

This means that CS system is almost(quasi-)integrable, that is

$$\dot{H}_n = e^{i\hat{U}}\dot{H}_n^0 e^{-i\hat{U}},$$

where $-2\dot{H}_n^0$ is the 2n-dimensional Laplacian $-2H_n^0$ restricted to $C^\infty(\mathbb{R}^{2n}\setminus\cup_{j,k}(x_j = x_k))$ and $\hat{U}$ is the operator of multiplication by $U(X_n)$,

$$U(X_n) = g\sum_{1\le k<j\le n} e_j e_k \phi(x_j - x_k), \qquad \phi(x_j - x_k) = \arctan\frac{x_j^2 - x_k^2}{x_j^1 - x_k^1}$$

As a result, there exists the simplest self-adjoint extension H_n of $\dot{H}_n$

$$H_n = e^{i\hat{U}}H_n^0 e^{-i\hat{U}}. \tag{2}$$

with the domain $D(H_n) = e^{i\hat{U}} D\left(H_n^0\right)$. Another self-adjoint extension of $\dot{H}_n$ is given by (2) in which, instead of H_n^0, another self-adjoint extension of $\dot{H}_n^0$ is considered.

The CS system of particles with different statistics has been studied by many authors [1, 2, 3, 4] since it can be derived (formally) in 3-d topological electrodynamics (its Lagrangian contains CS term) in the limit of the vanishing Maxwell term (the same is true for its relativistic version). There is a hope it can give a new mechanism of superconductivity, superfluidity and P, T violation.

2 Main result

Let $\Lambda \in \mathbb{R}^2$ be a compact set and assume the Dirichlet boundary conditions on the boundary $\partial\Lambda$. For the inverse temperature β, and the activity z_{e_s} of the particles with the charge e_s, the Gibbs (grand-canonical, equilibrium) RDMs are given by

$$\rho^\Lambda(X_m|Y_m) = Z_{(e)_m}\Xi_\Lambda^{-1}\sum_{n\geq 0}\prod_{s=1}^{n}\sum_{e'_s}\frac{z_{e'_s}^{n_s}}{n_s!}\int_{\Lambda^n} dX'_n \exp\{i[U(X_m,X'_n) - U(Y_m,X'_n)]\}P_{0(\Lambda)}^\beta(X_m,X'_n|Y_m,X'_n),$$

where Ξ_Λ is the grand partition function (it coincides with the numerator in the r.h.s. of this equality for the case $m = 0$, i.e. when there are no X_m and Y_m), $P_{0(\Lambda)}^\beta(X_m|Y_m)$ is the kernel of the semigroup, whose infinitesimal generator concides with $H_{n,\Lambda}^0$ ($-2H_{n,\Lambda}^0$ is the Laplacian in Λ^n with the Dirichlet boundary condition on the boundary $\partial\Lambda^n$), $Z_{(e)_m} = \prod_{s=1}^m z_{e_s}$, and the summation in e_s is performed over $E_{\{c\}}$.

$$P_{0(\Lambda)}^\beta(X_n|Y_n) = \prod_{j=1}^n P_{0(\Lambda)}^\beta(x_j|y_j), \qquad X_m = \left(X_m^1, X_m^2\right),\ Y_m = \left(Y_m^1, Y_m^2\right) \in \mathbb{R}^{2m}, \tag{3}$$

$P_{0(\Lambda)}^\beta(x|y)$ is the transition probability of the 2-dimensional free diffusion proccess with the Dirichlet boundary condition on $\partial\Lambda$.

$$P_{0(\Lambda)}^\beta(x|y) = \int P_{x,y}^\beta(d\omega)\chi_\Lambda(\omega),$$

$P_{x,y}(d\omega)$ is the conditional Wiener measure and $\chi_\Lambda(\omega)$ is the characteristic function of the paths that are inside Λ.

From the equality

$$U(X_m,X'_n) = U(X_m) + \sum_{j=1}^m\sum_{k=1}^n \phi(x_j - x'_k)e_je'_k + U(X'_n)$$

we obtain

$$\rho^\Lambda(X_m|Y_m) = Z_{(e)_m}\Xi_\Lambda^{-1}\exp\{i[U(X_m) - U(Y_m)]\}P_{0(\Lambda)}^\beta(X_m|Y_m)$$
$$\times\sum_{n\geq 0}\sum_{e'_s}\frac{z_{e'_s}^{n_s}}{n_s!}\int_{\Lambda^n} dX'_n\prod_{k=1}^n \exp\left\{i\sum_{j=1}^m e_je_k(\phi(x_j - x'_k) - \phi(y_j - x'_k))\right\}P_{0(\Lambda)}^\beta(x'_k|x'_k).$$

As a result

$$\rho^\Lambda(X_m|Y_m) = Z_{(e)_m}\exp\{i[U(X_m) - U(Y_m)] + G_\Lambda(X_m|Y_m)\}P_{0(\Lambda)}^\beta(X_m|Y_m), \tag{4}$$

$$G_\Lambda(X_m|Y_m) = \sum_e z_e \int\limits_\Lambda P^\beta_{0(\Lambda)}(x|x)\left[\exp\left\{i\sum_{j=1}^m ee_j(\phi(x_j-x)-\phi(y_j-x))\right\}-1\right]dx. \quad (5)$$

Here we used the equality

$$\Xi_\Lambda = \exp\left\{\sum_e z_e \int\limits_\Lambda P^\beta_{0(\Lambda)}(x|x)dx\right\}.$$

With the help of the equality

$$\exp\left\{i\arctan\frac{x^2}{x^1}\right\} = \frac{x}{|x|} = \left(\frac{x}{x^*}\right)^{\frac{1}{2}}, \qquad x = x^1 + ix^2,$$

we derive

$$\exp\left\{i\sum_{j=1}^m ee_j(\phi(x_j-x)-\phi(y_j-x))\right\} = \prod_{j=1}^m\left(\frac{(x-x_j)(x^*-y^*_j)}{(x^*-x^*_j)(x-y_j)}\right)^{\frac{1}{2}gee_j} = G_x(X_m|Y_m).$$

We have to use the Taylor expansions for $|x|<1$

$$(1-x)^g = 1 - gx + \frac{g(g-1)}{2}x^2 + \sum_{n\geq 3} C^g_n x^n = \sum_{n\geq 0} C^g_n x^n,$$

$$(1-x)^{-g} = 1 + gx + \frac{g(g+1)}{2}x^2 + \sum_{n\geq 3} C^{-g}_n x^n = \sum_{n\geq 0} C^{-g}_n x^n,$$

As a result for $g' = \frac{1}{2}gee_j$, $g' \notin \mathbb{Z}$, $\left|\frac{x_j}{x}\right| < 1$, $\left|\frac{y_j}{x}\right| < 1$

$$\left(\frac{x-x_j}{x^*-x^*_j}\right)^{g'} = \left(\frac{x}{x^*}\right)^{g'}\left(\frac{1-\frac{x_j}{x}}{1-\frac{x^*_j}{x^*}}\right)^{g'} = \left(\frac{x}{x^*}\right)^{g'}\left\{1 + g'\left(-\frac{x_j}{x}+\frac{x^*_j}{x^*}\right) + \frac{g'2}{2}\left(\frac{x^2_j}{x^2}+\frac{x^{*2}_j}{x^{*2}}\right)\right.$$

$$\left. + \frac{g'}{2}\left(\frac{x^{*2}_j}{x^{*2}} - \frac{x^2_j}{x^2}\right) - g'^2\left|\frac{x_j}{x}\right|^2 + \sum_{n^+_1 n^-_1 \geq 3} C^{-g'}_{n^+_1} C^{g'}_{n^-_1}\left(\frac{x^*_j}{x^*}\right)^{n^+_1}\left(\frac{x_j}{x}\right)^{n^-_1}\right\}.$$

Applying this formula we deduce

$$\left(\frac{(x-x_j)(x^*-y^*_j)}{(x^*-x^*_j)(x-y_j)}\right)^{g'} = 1 + G'_x(x_j|y_j)$$

$$+ \sum_{n^+_1+\cdots+n^-_2\geq 3} C^{-g'}_{n^+_1} C^{g'}_{n^-_1} C^{-g'}_{n^+_2} C^{g'}_{n^-_2}\left(\frac{x^*_j}{x^*}\right)^{n^+_1}\left(\frac{x_j}{x}\right)^{n^-_1}\left(\frac{y^*_j}{x^*}\right)^{n^-_2}\left(\frac{y_j}{x}\right)^{n^+_2},$$

where

$$G'_x(x_j|y_j) = g'\left(-\frac{x_j-y_j}{x}+\frac{x^*_j-y^*_j}{x^*}\right) - g'^2\left(\frac{x_j}{x}-\frac{x^*_j}{x^*}\right)\left(\frac{y_j}{x}-\frac{y^*_j}{x^*}\right)$$

$$+ \frac{g'^2}{2}\left(\frac{x^2_j+y^2_j}{x^2}+\frac{x^{*2}_j+y^{*2}_j}{x^{*2}}\right) + \frac{g'}{2}\left(\frac{y^2_j-x^2_j}{x^2}-\frac{y^{*2}_j-x^{*2}_j}{x^{*2}}\right) - g'^2\left[\left|\frac{x_j}{x}\right|^2+\left|\frac{y_j}{x}\right|^2\right].$$

As a result

$$G_x(X_m|Y_m) = 1 + G^0_{x,e}(X_m|Y_m) + \sum_{j=1}^{m} G'_x(x_j|y_j), \tag{6}$$

where

$$G^0_{x,e}(X_m|Y_m) = \sum_{n^+_{1,1}+\cdots+n^-_{2,m}\geq 3} \prod_{j=1}^{m} C^{-g'}_{n^+_{1,j}} C^{g'}_{n^-_{1,j}} C^{-g'}_{n^+_{2,j}} C^{g'}_{n^-_{2,j}} \left(\frac{x^*_j}{x^*}\right)^{n^+_{1,j}} \left(\frac{x_j}{x}\right)^{n^-_{1,j}} \left(\frac{y^*_j}{x^*}\right)^{n^-_{2,j}} \left(\frac{y_j}{x}\right)^{n^+_{2,j}}.$$

It can be checked that

$$-g'^2\left(\frac{x_j}{x} - \frac{x^*_j}{x^*}\right)\left(\frac{y_j}{x} - \frac{y^*_j}{x^*}\right) + \frac{g'^2}{2}\left(\frac{x_j^2+y_j^2}{x^2} + \frac{x^{*2}_j + y^{*2}_j}{x^{*2}}\right) - g'^2\left[\left|\frac{x_j}{x}\right|^2 + \left|\frac{y_j}{x}\right|^2\right]$$
$$= -g'^2\left|\frac{x_j - y_j}{x}\right|^2 + \frac{g'^2}{2}\left(\frac{(x_j-y_j)^2}{x^2} + \frac{(x^*_j - y^*_j)^2}{x^{*2}}\right). \tag{7}$$

This yields

$$G'_x(x_j|y_j) = -g'^2\left|\frac{x_j-y_j}{x}\right|^2 + G^-_x(x_j|y_j), \qquad G^-_x(x_j|y_j) = g'\left(-\frac{x_j-y_j}{x} + \frac{x^*_j - y^*_j}{x^*}\right)$$
$$+ \frac{g'^2}{2}\left(\frac{(x_j-y_j)^2}{x^2} + \frac{(x^*_j-y^*_j)^2}{x^{*2}}\right) + \frac{g'}{2}\left(\frac{y_j^2 - x_j^2}{x^2} - \frac{y^{*2}_j - x^{*2}_j}{x^{*2}}\right). \tag{8}$$

Let $l^+_m = \max(|x_j|, |y_j|, j = 1, \ldots, m)$. Let $\Lambda = B_L$ then

$$G_\Lambda(X_m|Y_m) = \sum_e z_e \left\{ \int\limits_{|x|\leq 2l^+_m} P^\beta_{0(\Lambda)}(x|x)\left[\prod_{j=1}^{m}\left(\frac{x^* - x^*_j}{x - x_j}\frac{x - y_j}{x^* - y^*_j}\right)^{\frac{1}{2}gee_j} - 1\right]dx \right.$$
$$\left. + \int\limits_{2l^+_m\leq|x|\leq L} P^\beta_{0(\Lambda)}(x|x) G^0_x(X_m|Y_m)dx + \int\limits_{2l^+_m\leq|x|\leq L} P^\beta_{0(\Lambda)}(x|x) G'_x(X_m|Y_m)dx \right\}. \tag{9}$$

For $|x| \geq 2l^+_m$ we have the bound

$$G^0_{x,e}(X_m|Y_m) \leq \frac{(2l^+_m)^3}{|x|^3} 2^{4|g'|m}.$$

Here we used the inequalities

$$\left|\left(\frac{x^*_j}{x^*}\right)^{n^+_{1,j}} \left(\frac{x_j}{x}\right)^{n^-_{1,j}} \left(\frac{y^*_j}{x^*}\right)^{n^-_{2,j}} \left(\frac{y_j}{x}\right)^{n^+_{2,j}}\right| \leq \frac{(2l^+_m)^3}{|x|^3} 2^{-(n^+_{1,j}+n^-_{1,j}+n^+_{2,j}+n^-_{2,j})},$$
$$\left|C^{+(-)g}_n\right| \leq C^{-|g|}_n > 0.$$

After applying them we enlarge the sum in the expression for $G^0_{x,e}$ to the sum over $(\mathbb{Z}^+)^{4m}$.

Since $P_\Lambda(x|x)$ tends to $(2\pi\beta)^{-1}$ the first and the second terms in (9) have limits when L tends to infinity. We have only to calculate the third term. Let us show that

$$\int\limits_{r\leq|x|\leq L} G^-_x(x'|y')dx = 0. \tag{10}$$

For arbitrary r, L, $v = v^1 + iv^2$, $B = B_L \backslash B_r$ we have

$$\int\limits_B \left(\frac{v}{x} - \frac{v^*}{x^*}\right) dx = -2i \left[v^1 \int\limits_B \frac{x^2}{|x|^2} dx - v^2 \int\limits_B \frac{x^1}{|x|^2} dx \right] = 0,$$

$$\int\limits_B \left(\frac{v}{x^2} + \frac{v^*}{x^{*2}}\right) dx = 2 \left\{ v^1 \int\limits_B \frac{\left(x^1\right)^2 - \left(x^2\right)^2}{|x|^4} dx + 2v^2 \int\limits_B \frac{x^1 x^2}{|x|^4} dx \right\} = 0,$$

$$\int\limits_B \left(\frac{v}{x^2} - \frac{v^*}{x^{*2}}\right) dx = 2i \left\{ v^2 \int\limits_B \frac{\left(x^1\right)^2 - \left(x^2\right)^2}{|x|^4} dx - 2v^1 \int\limits_B \frac{x^1 x^2}{|x|^4} dx \right\} = 0.$$

All the above integrals are zero since the all the functions change signs when either a sign of one of the variables is changed, or a permutation is done.

As a result

$$\int\limits_{r<|x|\le L} G'_x(X_m|Y_m) dx = -\frac{1}{4} g^2 \left(\sum_e z_e e^2\right) N_{r,L} \sum_{j=1}^m e_j^2 |x_j - y_j|^2. \tag{11}$$

The integral in the right-hand-side of this equality diverges as $2\ln L$. For $g' = k$, $k \in \mathbb{Z}$ we obtain (6) in which we have to put $C_n^k = 0$ for $n > k$, $C_n^k = \frac{k!}{(n-k)!n!}$.

From (9), (11) we derive

$$G_\Lambda(X_m|Y_m) = \sum_e z_e \left\{ \int\limits_{|x|\le 2l_m^+} P^\beta_{0(\Lambda)}(x|x) \left[\prod_{j=1}^m \left(\frac{x^* - x_j^*}{x - x_j} \frac{x - y_j}{x^* - y_j^*} \right)^{\frac{1}{2} gee_j} - 1 \right] dx \right.$$

$$\left. + \int\limits_{2l_m^+ \le |x| \le L} P^\beta_{0(\Lambda)}(x|x) G^0_x(X_m|Y_m) dx - \frac{1}{4} g^2 \left(\sum_e z_e e^2\right) N_{2l_m^+,L} \sum_{j=1}^m e_j^2 |x_j - y_j|^2 \right\}. \tag{12}$$

Using the equalities

$$\lim_{L\to\infty} P^\beta_{0(B_L)}(x|x) = (2\pi\beta)^{-1}, \qquad \lim_{L\to\infty} (\ln L)^{-1} N_{r,L} = \beta^{-1}$$

and the fact that $G^0_{x,e}$ is an integrable function in $\Lambda\backslash\{0\}$ and we derive the following result.

Theorem 1. *Let Λ coincide with B_L.*

I. If $g^2 = g_0^2 (\ln L)^{-1}$ then

$$\lim_{L\to\infty} \rho^{B_L}(X_m|Y_m) = Z_{(e)_m} \exp\left\{i[U(X_m) - U(Y_m)]\right\} P^\beta_0(X_m|Y_m)$$

$$\times \exp\left\{ -\frac{1}{4\beta} g_0^2 \left(\sum_e z_e e^2\right) \sum_{j=1}^m e_j^2 |x_j - y_j|^2 \right\}.$$

II. If $\lim_{L\to\infty} g^2 \ln L = 0$ then

$$\lim_{L\to\infty} \rho^{B_L}(X_m|Y_m) = Z_{(e)_m} \exp\{i[U(X_m) - U(Y_m)]\} P^\beta_0(X_m|Y_m).$$

III. If $\lim_{L\to\infty} g^2 \ln L = \infty$ then

$$\lim_{L\to\infty} \rho^{B_L}(X_m|Y_m) = 0, \qquad x_j \ne y_j.$$

The mean-field type limit for the quantum CS system does not exit contrary to the classical CS particle system [8]. But there is a similarity in the behavior of the RDMs in the weak coupling limit and the Gibbs correlation functions in the mean-field type limit.

Description of integrable systems with magnetic interaction in the thermodynamic limit can be found in [5, 6, 7, 9, 10].

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