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Semiclassically Concentrates Waves for the Nonlinear Schrödinger Equation with External Field

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Classes of solutions, asymptotic in small parameter $\hbar$, $\hbar \to 0$, are constructed to the generalized nonlinear Schrödinger equation (NSE) in a multi-dimensional space with an external field in the framework of the WKB-Maslov method. Asymptotic semiclassically concentrated solutions (SCS), regarded as multi-dimensional solitary waves, are introduced for the NSE with an external field and cubic local nonlinearity. The one-dimensional soliton dynamics in an external field of a special form is discussed. Another class of asymptoitic SCS solutions is constructed for the NSE with Gaussian non-local potential and a local external field. These solutions are similar to the trajectory-coherent states or squeezed states in quantum mechanics.

1 Introduction

We study soliton-like properties of nonintegrable generalizations of the nonlinear Schrödinger equation (NSE)

$$\left\{-i\hbar\partial_t + \hat{\mathcal{H}}\left(t, |\Psi|^2\right)\right\}\Psi = 0 \tag{1}$$

within the framework of the semiclasical WKB-Maslov method [1]. Here, $\Psi = \Psi(\vec{x}, t)$ is a complex smooth function, $\vec{x} \in \mathbb{R}^n$, $t \in \mathbb{R}^1$; $|\Psi|^2 = \Psi^*\Psi$, Ψ^* is the function complex conjugate of Ψ; $\hat{\mathcal{H}}\left(t, |\Psi|^2\right)$ is a nonlinear operator, $\partial_t\Psi = \partial\Psi/\partial t$. The Planck's constant $\hbar$ plays the role of an asymptotic parameter.

Equation (1) arises in the statistical physics and quantum theory of condensed matter [2]. The evolution of bosons is described in terms of the secondary quantized Schrödinger equation. In Hartree's approximation it leads to the classical multi-dimensional Schrödinger equation with a non-local nonlinearity for one-particle functions, i.e. a Hartree type equation. The special case of equation (1), the NSE with local cubic nonlinearity

$$i\hbar\Psi_{,t} + \frac{\hbar^2}{2}\Psi_{,xx} + 2g|\Psi|^2\Psi = 0, \tag{2}$$

is used, in particular, in nonlinear optics (see, for example, [3, 4]). Here $\Psi = \Psi(x, t)$, $x \in \mathbb{R}^1$, g is a real nonlinearity parameter, $\Psi_{,t} = \partial\Psi/\partial t$, $\Psi_{,x} = \partial\Psi/\partial x$.

Equation (2) is integrated by the Inverse Scattering Transform (IST) method and has soliton solutions [5]. Solitons are localized wave packets propagating without distortion and interacting elastically in mutual collisions. The soliton conception is of commonly used in various fields of nonlinear physics and mathematics (see [6, 7, 8] and Refs. herein).

A fairly wide class of nonlinear equations, nonintegrable via the IST method, was found to possess soliton-like solutions. They are concentrated in a sense and conserve this property in the course of evolution. These solutions are referred to as solitary waves (SWs), quasi-solitons, etc. There is a large number of papers studying SWs. For example, so called squeezed (compressed) light states and the important problem of the correspondence between the stressed states describing the quantum properties of a radiation and the optical solitons are analyzed in [11] in terms of NLS-solitons. Systematic study of soliton excitations in molecular systems was carried out by Davidov [9] and was continued in subsequent works [10].

Note that in the optical pulse propagation theory the function Ψ is an envelope of the electromagnetic field that is quite different from the quantum mechanical meaning of Ψ. Though, in both cases Ψ is a square-integrable function which norm is conserved. This can be considered as a ground to apply quantum mechanical ideas and methods to the pulse propagation theory. The semiclassical approach in this case implies that we deal with narrow wave packets, and the asymptotic small parameter $\hbar$ is a characteristic of the packet width.

Soliton properties in nonintegrable systems can be investigated either using computer simulations or by approximate methods.

We construct asymptotic semiclassically concentrated solutions, regarded as multi-dimensional solitary waves, for the NSE with cubic local nonlinearity in the presence of an external field. The one-dimensional soliton dynamics in the external field of a special form is discussed in terms of the asymptotic SCS as an illustration.

Another class of the SCS is introduced and studied for the NSE with non-local unitary nonlinearity, the Hartree type equation. This class of solutions is similar to the trajectory-coherent states or squeezed states in quantum mechanics. A class of such solutions, asymptotic in small parameter $\hbar$ ($\hbar \to 0$), is constructed for the one-dimensional Hartree type equation with Gaussian non-local potential.

2 The nonlinear Schrödinger equation with external field

The generalized NSE with cubic local nonlinearity is written as follows [2, 7, 9]:

$$\left\{-i\hbar\partial/\partial t + \frac{1}{2}(-i\hbar\nabla - \vec{\mathcal{A}}(\vec{x},t))^2 + u(\vec{x},t) - 2g|\Psi(\vec{x},t)|^2\right\}\Psi(\vec{x},t) = 0. \tag{3}$$

Here $u(\vec{x},t)$, $\vec{\mathcal{A}}(\vec{x},t)$ are given functions determining an external field; g is a real parameter of nonlinearity.

The key moment of the asymptotic method is choice of a class of functions singularly depending on the asymptotic parameter in which asymptotic solutions are constructed.

To define soliton-like asymptotic solutions to (3) we need some auxiliary notions. Let $\hat{\vec{x}}(=\vec{x})$ and $\hat{\vec{p}}(=-i\hbar\nabla)$ are the position and momentum operators, respectively, with the commutators

$$[\hat{x}_k,\hat{p}_s] = i\hbar\delta_{k,s}, \qquad [\hat{x}_k,\hat{x}_s] = [\hat{p}_k,\hat{p}_s] = 0, \qquad k,s=\overline{1,n}.$$

A smooth function $A(t,\vec{x},\vec{p})$ of t and of real vector variables $\vec{x}$ and $\vec{p}$ is a symbol of the (Weyl) operator $\hat{A}(t,\vec{x},\hat{\vec{p}})$.

The mean value of the operator $\hat{A}$ by a function $\Psi(\vec{x},t,\hbar)$ is defined as

$$\langle A\rangle = \langle\Psi|\hat{A}|\Psi\rangle/\|\Psi\|^2, \qquad \|\Psi\|^2 = \langle\Psi|\Psi\rangle = \int_{\mathbb{R}^n}|\Psi|^2 d\vec{x}, \tag{4}$$

$$\langle\Psi|\hat{A}(t)|\Psi\rangle = \int_{\mathbb{R}^n}\Psi^*(\vec{x},t)\hat{A}(t)\Psi(\vec{x},t)d\vec{x}.$$

For the operators $\vec{x}$, $\hat{\vec{p}}$ we have $\langle\vec{x}\rangle = \vec{X}(t,\hbar)$, $\langle\vec{p}\rangle = \vec{P}(t,\hbar)$. We assume that there exist the limits $\lim\limits_{\hbar\to 0}\vec{X}(t,\hbar) = \vec{X}(t)$, $\lim\limits_{\hbar\to 0}\vec{P}(t,\hbar) = \vec{P}(t)$. The $2n$-vector function $Z(t) = \{\vec{X}(t),\vec{P}(t),\ 0\le t\le T\}$ is referred to as the phase orbit corresponding to the function $\Psi(\vec{x},t)$.

Let $\mathbb{CS}^\hbar(Z(t)) \equiv \mathbb{CS}^\hbar$ be the class of semiclassically concentrated functions associated with an arbitrary phase orbit $Z(t)$ as follows.

Definition 1. A function $\Psi(\vec{x},t)$ belongs to the class $\mathbb{CS}^\hbar$

(i) if there exists the limit

$$\lim_{\hbar\to 0} |\Psi(\vec{x},t,\hbar)|^2/\|\Psi\|^2 = \delta(\vec{x}-\vec{X}(t)),$$

(ii) there exist the centered moments of arbitrary order with respect to $\vec{X}(t)$, $\vec{P}(t)$.

A solution $\Psi(\vec{x},t,\hbar)$ of (3), $\Psi \in \mathbb{CS}^\hbar$, is called the semiclassically concentrated solution (SCS).

It was proved in Ref. [12] that if $\Psi(\vec{x},t,\hbar)$ is a semiclassically concentrated solution of (3), then $Z(t) = \{\vec{X}(t), \vec{P}(t)\}$ is a solution of the classical Hamilton system with the Hamiltonian $\mathcal{H}_{\rm cl}(\vec{p},\vec{x},t) = \frac{1}{2}(\vec{p}-\vec{\mathcal{A}}(\vec{x},t))^2 + u(\vec{x},t)$.

Let us denote by $\mathcal{Q}^t_\hbar$ a class of semiclassically concentrated functions $\Psi(\vec{x},t,\hbar)$ singularly depending on the asymptotic parameter $\hbar$, $\hbar\to 0$,

$$\mathcal{Q}^t_\hbar = \left\{\Psi(\vec{x},t,\hbar) : \Psi(\vec{x},t,\hbar) = \rho(\theta,\vec{x},t,\hbar)\exp\left[\frac{i}{\hbar}S(\vec{x},t,\hbar)\right]\right\}. \tag{5}$$

Here $\theta = \hbar^{-1}\sigma(\vec{x},t,\hbar)$ is a "fast" variable; $\sigma(\vec{x},t,\hbar)$, $\rho(\theta,\vec{x},t,\hbar)$, and $S(\vec{x},t,\hbar)$ are real functions regular in $\hbar$, that is $S(\vec{x},t,\hbar) = S^{(0)}(\vec{x},t) + \hbar S^{(1)}(\vec{x},t) + \cdots$.

The class $\mathcal{Q}^t_\hbar$ can be considered as a generalization of the solitary wave since the one-soliton solution for the NSE (2) belongs to the $\mathcal{Q}^t_\hbar$. Note that the derivative operators $\partial/\partial t$ and ∇ are extended in acting on the functions of the class (5):

$$-i\hbar\partial/\partial t = -i\hbar\partial/\partial t\Big|_{\theta=\rm const} - i\sigma_{,t}\partial/\partial\theta, \qquad -i\hbar\nabla = -i\hbar\nabla|_{\theta=\rm const} - i(\nabla\sigma)\partial/\partial\theta,$$

where $\sigma_{,t} = \partial\sigma/\partial t$. In what follows we put

$$\partial/\partial t|_{\theta=\rm const} \equiv \partial_t, \qquad \nabla|_{\theta=\rm const} \equiv \nabla, \qquad \partial/\partial\theta \equiv \partial_\theta. \tag{6}$$

Let us set estimates for these operators.

Definition 2. An operator $\hat{A}$ has the asymptotic estimate $\hat{O}(\hbar^\alpha)$ on the class $\mathcal{Q}^t_\hbar$, $\hat{A} = \hat{O}\left(\hbar^\alpha\right)$, if $\forall\ \Psi \in \mathcal{Q}^t_\hbar$ the asymptotic estimate

$$\|\hat{A}\Psi\|/\|\Psi\| = O\left(\hbar^\alpha\right), \qquad \hbar\to 0, \tag{7}$$

is valid.

Note that similar estimates are also valid for mean values of operators,

$$|\langle\Psi|\hat{A}|\Psi\rangle|/\|\Psi\| = O\left(\hbar^\alpha\right), \qquad \hbar\to 0.$$

For the derivative operators (6) we have

$$i\hbar\partial_t + S_{,t} = \hat{O}(\hbar), \qquad i\hbar\nabla + \nabla S = O(\hbar), \qquad \vec{x} = O(1), \qquad \partial_\theta = O(1). \tag{8}$$

These estimates permits us to construct a solution of equation (3) in the form of asymptotic series in $\hbar$.

When studying the asymptotic solution, the leading term is of primary interest. So, we construct the asymptotic SCS to equation (3) in the class $\mathcal{Q}^t_\hbar$ with an accuracy of $O\left(\hbar^2\right)$.

To this end we substitute the function $\Psi(\vec{x},t)$ of the form (5) into (3), gather and sum both $\hbar$–free terms and terms of the power $\hbar^1$, and put every of these sums to zero. Note that the residual has the estimates $O\left(\hbar^2\right)$. Next, we separate the equations for the function ρ with the "fast" variable θ from the other equations and solve them under the constraint $\lim\limits_{\theta\to\infty}\rho(\theta,\vec{x},t,\hbar)=\lim\limits_{\theta\to\infty}\rho_{,\theta}(\theta,\vec{x},t,\hbar)=0$.

As a result the asymptotic solution taken with the accuracy of $O\left(\hbar^2\right)$ is of the form

$$\Psi=\Psi_0(\theta,\vec{x},t,\hbar)[1+\hbar(w(\theta,\vec{x},t)+iv(\theta,\vec{x},t))]+O\left(\hbar^2\right), \tag{9}$$

where

$$\Psi_0=\rho(\theta,\vec{x},t,\hbar)\exp\left[\frac{i}{\hbar}(S^{(0)}(\vec{x},t)+\hbar S^{(1)}(\vec{x},t))\right], \tag{10}$$

$$\theta=\frac{1}{\hbar}\sigma^{(0)}(\vec{x},t)+\sigma^{(1)}(\vec{x},t), \tag{11}$$

$$\rho=\sqrt{\frac{\left(\nabla\sigma^{(0)}\right)^2}{2g}}\cosh^{-1}\theta,\quad g>0. \tag{12}$$

Here, $S^{(0)}$, $S^{(1)}$, $\sigma^{(0)}$, $\sigma^{(1)}$ are real functions of $\vec{x}$ and t independent from $\hbar$ which are determined by the following system:

$$S^{(0)}_{,t}+u+\frac{1}{2}\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right)^2=\frac{1}{2}\left(\nabla\sigma^{(0)}\right)^2, \tag{13}$$

$$\sigma^{(0)}_{,t}+\langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right),\nabla\sigma^{(0)}\rangle=0, \tag{14}$$

$$S^{(1)}_{,t}+\langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right),\nabla S^{(1)}\rangle-\langle\nabla\sigma^{(0)},\nabla\sigma^{(1)}\rangle+\frac{\nu}{2}\langle\nabla\sigma^{(0)},\nabla\rangle\ln\frac{\left(\nabla\sigma^{(0)}\right)^2}{g}+\frac{\nu}{2}\Delta\sigma^{(0)}=0, \tag{15}$$

$$\sigma^{(1)}_{,t}+\langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right),\nabla\sigma^{(1)}\rangle+\langle\nabla\sigma^{(0)},\nabla S^{(1)}\rangle-\frac{\nu}{2}\left[\left(\ln\frac{\left(\nabla\sigma^{(0)}\right)^2}{g}\right)_{,t}+\langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right),\nabla\ln\frac{\left(\nabla\sigma^{(0)}\right)^2}{g}\rangle+\langle\nabla,\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right)\rangle\right]=0. \tag{16}$$

Here, $\nu=\operatorname{sign}(\theta)$ and $\langle\vec{a},\vec{b}\rangle$ denotes the Euclidean scalar product of the vectors: $\sum\limits_{j=1}^{n}a_jb_j$.

The functions $w(\theta,\vec{x},t)$, $v(\theta,\vec{x},t)$ are written as

$$\rho(\theta,\vec{x},t)w(\theta,\vec{x},t)=\sqrt{\frac{2}{g\left(\nabla\sigma^{(0)}\right)^2}}\frac{1}{\cosh\theta}\left\{c_1(\vec{x},t)\tanh\theta+\frac{1}{2}\langle\nabla\sigma^{(0)},\nabla\sigma^{(1)}\rangle+\frac{1}{12}\left[\Delta\sigma^{(0)}+\langle\nabla\sigma^{(0)},\nabla\ln\frac{\left(\nabla\sigma^{(0)}\right)^2}{g}\rangle\right]\left(\sinh\theta\cosh\theta-\nu\cosh^2\theta\right)\right\}, \tag{17}$$

$$\rho(\theta,\vec{x},t)v(\theta,\vec{x},t)=\sqrt{\frac{2}{g^2\left(\nabla\sigma^{(0)}\right)^2}}\left\{\frac{c_1(\vec{x},t)}{\cosh\theta}+\frac{1}{4}\left[\langle\nabla,\nabla S^{(0)}-\vec{\mathcal{A}}\rangle+\left(\partial_t+\langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right),\nabla\rangle\right)\ln\frac{(\nabla\sigma^{(0)})^2}{g}\right](\nu\sinh\theta-\cosh\theta)\right\}. \tag{18}$$

Here, a function $c_1(\vec{x},t)$ is determined by successive approximations.

3 One-dimensional NSE-soliton in external field

To assess an efficacy of the asymptotic approach it is of interest to compare the asymptotic results with a well known problem. To this end let us apply the above asymptotic solution to the one-dimensional nonlinear Schrödinger equation with an external field $u(x, t)$ that is read as

$$i\hbar\Psi_{,t} + \frac{\hbar^2}{2}\Psi_{,xx} + 2g|\Psi|^2\Psi - u\Psi = 0. \tag{19}$$

In accordance with (9)–(12) soliton-like asymptotic solution for equation (19) is

$$\Psi = \sqrt{\frac{(\sigma^{(0)}_{,x})^2}{2g}}\ \exp\left[\frac{i}{\hbar}\left(S^{(0)}(x,t) + \hbar S^{(1)}(x,t)\right)\right]\cosh^{-1}\theta. \tag{20}$$

Here, $S^{(0)}$, $\sigma^{(0)}$, $S^{(1)}$, $\sigma^{(1)}$ are functions of x and t, independent of $\hbar$. Equations (13)–(16) takes the form

$$S^{(0)}_{,t} + \frac{1}{2}\left(S^{(0)}_{,x}\right)^2 + u = \frac{1}{2}\left(\sigma^{(0)}_{,x}\right)^2, \qquad \sigma^{(0)}_{,t} + S^{(0)}_{,x}\sigma^{(0)}_{,x} = 0, \tag{21}$$

$$S^{(1)}_{,t} + S^{(0)}_{,x}S^{(1)}_{,x} - \sigma^{(0)}_{,x}\sigma^{(1)}_{,x} + \frac{\nu}{2}\sigma^{(0)}_{,x}\left(\ln\frac{(\sigma^{(0)}_{,x})^2}{g}\right)_{,x} + \frac{\nu}{2}\sigma^{(0)}_{,xx} = 0, \tag{22}$$

$$\sigma^{(1)}_{,t} + S^{(0)}_{,x}\sigma^{(1)}_{,x} + \sigma^{(0)}_{,x}S^{(1)}_{,x} = \frac{\nu}{2}\left(\ln\frac{(\sigma^{(0)}_{,x})^2}{g}\right)_{,t} + \frac{\nu}{2}S^{(0)}_{,x}\left(\ln\frac{(\sigma^{(0)}_{,x})^2}{g}\right)_{,x} + \frac{\nu}{2}\left(S^{(0)}\right)_{,xx}. \tag{23}$$

At $u = 0$ the functions

$$S^{(0)} = 2(\eta^2 - \xi^2)t + 2\xi x + \varphi_0, \tag{24}$$

$$\sigma^{(0)} = -4\xi\eta t + 2\eta(x - x_0), \tag{25}$$

$$S^{(1)} = \sigma^{(1)} = 0. \tag{26}$$

satisfy the system (21)–(23) and determine the *exact* one-soliton solution to the nonlinear Schrödinger equation (19) in the form (20). Here, constants ξ, η, φ_0, x_0 are soliton parameters: 2ξ is a velocity, η is related to an amplitude, φ_0 is an initial phase, x_0 is an initial soliton position.

Let us construct the asymptotic solution of the form (20) so that it turns into the exact one-soliton solution at $u \to 0$. We will refer to this asymptotic solution as *asymptotic soliton* for equation (19).

In accordance with (24)–(26) we take the solutions of equations (21) as

$$\begin{aligned} S^{(0)} &= 2\left(\eta^2 - \xi^2\right)t + 2\xi x + \varphi_0 + h(x,t), \\ \sigma^{(0)} &= -4\xi\eta t + 2\eta(x - x_0) + f(x,t). \end{aligned} \tag{27}$$

Then for functions h and f we have

$$\begin{aligned} &h_{,t} + \frac{1}{2}\left(4\xi h_{,x} + h^2_{,x}\right) + u = \frac{1}{2}\left(4\eta f_{,x} + f^2_{,x}\right), \\ &f_{,t} + (2\xi + h_{,x})f_{,x} + 2\eta h_{,x} = 0. \end{aligned} \tag{28}$$

Taking f as $f(x,t) = -2\eta x + 4\eta\xi t + w(x,t)$, we obtain

$$\sigma^{(0)} = -2\eta x_0 + w(x,t). \tag{29}$$

Equations (21), (28) result in the following equations for the functions h and w:

$$h_{,t} + \frac{1}{2}\left(4\xi h_{,x} + h_{,x}^2\right) + u = \frac{1}{2}\left(-4\eta^2 + w_{,x}^2\right), \tag{30}$$

$$w_{,t} + (2\xi + h_{,x})w_{,x} = 0. \tag{31}$$

For $h_{,x} = h_{,x}(x)$ the characteristic equation of (31), $dx/dt = 2\xi + h_{,x}$, has a special solution as an arbitrary function $w = w(z)$ of the variable $z = t - \int(2\xi + h_{,x})^{-1}dx$. Then with the change of variables $(x,t) \to (x,z)$ (22), (23) are simplified as

$$\frac{w'(z)}{2\xi + h_{,x}}\left(\sigma_{,x}^{(1)} - \frac{1}{2\xi + h_{,x}}\sigma_{,z}^{(1)}\right) + (2\xi + h_{,x})S_{,x}^{(1)} + \frac{3\nu}{2}\frac{w''(z) + w'(z)h_{,xx}}{(2\xi + h_{,x})^2} = 0, \tag{32}$$

$$\frac{w'(z)}{2\xi + h_{,x}}\left(S_{,x}^{(1)} - \frac{1}{2\xi + h_{,x}}S_{,z}^{(1)}\right) - (2\xi + h_{,x})\sigma_{,x}^{(1)} = \frac{\nu}{2}h_{,xx}. \tag{33}$$

Had we chosen a special solution of equations (32), (33) in the form $w(z) = \alpha z$, $\alpha = \text{const}$, then the functions $\sigma^{(1)} \equiv m(x)$ and $S^{(1)} \equiv n(x)$ are dependent on x only and are determined by the equations

$$\frac{\alpha}{2\xi + h_{,x}}m'(x) + (2\xi + h_{,x})n'(x) + \frac{3\nu}{2}\frac{\alpha h_{,xx}}{(2\xi + h_{,x})^2} = 0,$$

$$\frac{\alpha}{2\xi + h_{,x}}n'(x) - (2\xi + h_{,x})m'(x) = \frac{\nu}{2}h_{,xx}.$$

The potential u according to (30) reads

$$u = \frac{1}{2}\cdot\frac{\alpha^2}{(2\xi + h_{,x})^2} - \frac{1}{2}(2\xi + h_{,x})^2 + 2\left(\xi^2 - \eta^2\right). \tag{34}$$

Let us take into account that the velocity V of the exact one-soliton solution of the NSE (19) at $u = 0$ is equal to $V = 2\xi$. In terms of the "fast" variable $\theta = (2\eta/\hbar)(x - x_0 - 2\xi t)$ it will be

$$V = -\frac{\partial\theta}{\partial t}\Big/\frac{\partial\theta}{\partial x}. \tag{35}$$

For the considered asymptotic solution

$$\theta = \frac{1}{\hbar}\sigma^{(0)} + \sigma^{(1)} = \frac{\alpha}{\hbar}\left(t - \int\frac{dx}{2\xi + h_{,x}}\right) - \frac{2\eta x_0}{\hbar} + m(x), \tag{36}$$

and, with respect to (35), we have

$$V = V(x) = (2\xi + h_{,x})\left[1 - \frac{\hbar}{\alpha}m'(x)(2\xi + h_{,x})\right]^{-1}. \tag{37}$$

Note that at $\hbar \to 0$ we obtain $V \to (2\xi + h_{,x})$. The function

$$V_0(x) = 2\xi + h_{,x} \tag{38}$$

has the meaning of the velocity (at $\hbar \to 0$) of the asymptotic soliton moving in the external field $u(x)$. From (34) it follows

$$\lim_{h(x)\to 0} u(x) = \frac{\alpha^2}{8\xi^2} - 2\eta^2.$$

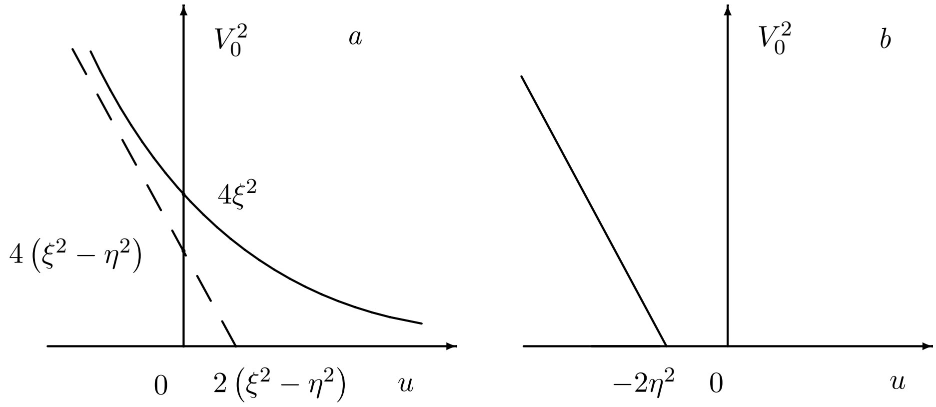

Figure 1.

For $u(x) \to 0$ at $h_{,x} \to 0$ one needs to put $\alpha = \pm 4\xi\eta$. If we take $\alpha = -4\xi\eta$ then the potential u according to (34) and (38) becomes

$$u(x) = \frac{8\xi^2\eta^2}{V_0(x)^2} - \frac{1}{2}V_0(x)^2 + 2\left(\xi^2 - \eta^2\right). \tag{39}$$

Solving (39) with respect to V_0^2 we obtain

$$V_0(x)^2 = -u(x) - 2(\eta^2 - \xi^2) + \sqrt{\left[u(x) + 2\left(\eta^2 - \xi^2\right)\right]^2 + 16\eta^2\xi^2}. \tag{40}$$

Note that in (40) we are to take the positive value of the square root and $V_0^2 \to 4\xi^2$ at $u \to 0$.

The general form of the function $V_0^2(u)$ is shown as in Fig. 1 at $\xi \neq 0$ (a) and at $\xi = 0$ (b). It can be seen that the potential well ($u \leq 0$) increases the soliton velocity and the potential barrier ($u \geq 0$) monotonously decreases it with respect to the free soliton velocity equal to 2ξ without a barrier reflection. The last feature is the nonlinearity effect.

Let us collect the expressions determining the asymptotic one-soliton solution (20) for equation (19) with the external field $u(x)$.

Equations (27), (29) for the functions $S^{(0)}$, $\sigma^{(0)}$ are written as

$$S^{(0)} = 2\left(\eta^2 - \xi^2\right)t + \int_{-\infty}^{x} (V_0(y) - 2\xi)dy + 2\xi x + \varphi_0,$$

$$\sigma^{(0)} = 4\xi\eta\left(\int_{-\infty}^{x}\left(\frac{1}{V_0(y)} - \frac{1}{2\xi}\right)dy - t\right) + 2\eta(x - x_0).$$

The functions $S^{(1)} \equiv n(x)$, $\sigma^{(1)} \equiv m(x)$ are given by

$$\sigma^{(1)\prime} = m'(x) = -\frac{\nu V_{0,xx}(x)}{2D}\left(\frac{48\xi^2\eta^2}{V_0(x)^3} + V_0(x)\right),$$

$$S^{(1)\prime}(x) = n'(x) = \frac{4\nu\xi\eta V_{0,xx}(x)}{V_0(x)D}, \qquad D = \frac{16\xi^2\eta^2}{V_0^2(x)} + V_0^2(x).$$

The "fast" variable θ (29) takes the form

$$\theta = \frac{4\xi\eta}{\hbar}\left(\int_{-\infty}^{x}\left(\frac{1}{V_0(y)} - \frac{1}{2\xi}\right)dy - t\right) + 2\frac{\eta}{\hbar}(x - x_0) + m(x),$$

the phase

$$\Phi = \frac{1}{\hbar}S^{(0)} + S^{(1)} = \frac{2}{\hbar}\left(\eta^2 - \xi^2\right)t + \frac{1}{\hbar}\int_{-\infty}^{x}(V_0(y) - 2\xi)dy + \frac{2\xi x + \varphi_0}{\hbar} + n(x).$$

The velocity $V(x)$ of the asymptotic soliton in the external field $u(x)$ with respect to (37) is

$$V(x) = V_0(x)\left[1 + \frac{\hbar}{4\xi\eta} m'(x)V_0(x)\right]^{-1}.$$

4 The Hartree type equation

The asymptotic approach appears to be more effective for the NSE with non-local nonlinearity, the Hartree type equation (HTE). A construction of asymptotic solution to the multi-dimensional HTE with external field and unitary non-local nonlinearity in terms of the WKB-Maslov method is developed in [13]. Here we consider the one-dimensional HTE with Gaussian non-local potential

$$\left\{-i\hbar\partial_t + \mathcal{H}(\hat{p}, x, t) + \hat{g}V_0 \int_{-\infty}^{+\infty} dy\, \exp\left[\frac{-(x-y)^2}{2\gamma^2}\right] \frac{|\Psi(y,t)|^2}{\|\Psi\|^2}\right\}\Psi = 0, \tag{41}$$

where $\mathcal{H}(p,x,t) = \frac{p^2}{2m} + u(x,t)$, $u(x,t) = \frac{1}{2}kx^2 + lx$ is the Hamiltonian of an effective particle in the external field that is the sum of an oscillator field and a stationary homogeneous field. Note that $\hat{g} = g\|\Psi\|^2$ is assumed to be $O(1)$ and k, V_0, l are real parameters.

The HTE is not solvable by the IST method even in one-dimensional case. To define a class of semiclassically concentrated functions similar to (5) we turn to the quantum mechanics where functions of this type are well known coherent and “squeezed” states (see, for example, [14, 15]).

Following to these ideas, consider a class of functions $\mathcal{P}_\hbar^t$ in which we will find asymptotic solutions of equation (41), it as

$$\mathcal{P}_\hbar^t = \left\{\Psi : \Psi(x,t,\hbar) = \varphi\left(\frac{\Delta x}{\sqrt{\hbar}}, t, \hbar\right)\exp\left[\frac{i}{\hbar}(S(t) + P(t)\Delta x)\right]\right\}. \tag{42}$$

Here the function $\varphi(\xi, t, \hbar)$ belongs to the Schwartz space $\mathbb{S}$ in variable $\xi \in \mathbb{R}^1$ and depends smoothly on t and regularly on $\sqrt{\hbar}$ for $\hbar \to 0$. We assume here that $\Delta x = x - X(t)$; the real function $S(t)$ and the 2-dimensional vector function $Z(t) = (P(t), X(t))$, which characterize the class $\mathcal{P}_\hbar^t$, are indepent of $\hbar$ and are to be determined. More general case when S, P, X are regular functions of $\sqrt{\hbar}$ is considered in [13]. The functions of the class $\mathcal{P}_\hbar^t$ are normalized to $\|\Psi(t)\|^2 = \langle\Psi(t)|\Psi(t)\rangle$ in the space $L_2(\mathbb{R}_x^1)$ with the norm (4).

In addition, let us define the following class of functions

$$\mathcal{C}_\hbar^t = \left\{\Psi : \Psi(x,t,\hbar) = \varphi\left(\frac{\Delta x}{\sqrt{\hbar}}, t\right)\exp\left[\frac{i}{\hbar}(S(t,\hbar) + \langle P(t,\hbar), \Delta x\rangle)\right]\right\}, \tag{43}$$

where the functions $\varphi(\xi, t)$, as distinct from (42), are independent of $\hbar$.

At any fixed point in time $t \in \mathbb{R}^1$, the functions of the class $\mathcal{P}_\hbar^t$ *are concentrated*, in the limit of $\hbar \to 0$, in a neighborhood of a point lying on the phase curve $z = Z(t, 0)$, $t \in \mathbb{R}^1$ [13] and are referred to as *trajectory-concentrated functions* (TCF).

In definition of the class of the TCF the phase trajectory $Z(t,\hbar)$ and the scalar function $S(t,\hbar)$ are free “parameters”. Note that for a linear Schrödinger equation, $g = 0$, the class $\mathcal{P}_\hbar^t$ includes the well-known dynamic (compressed) coherent states of quantum systems with quadratic Hamiltonians (see for details [16]).

Let us consider principal moments of the asymptotic solution construction for equation (41) in the class $\mathcal{P}_\hbar^t$ (see for details [12]).

Consider functions Φ of the class $\hat{\mathcal{P}}^t_\hbar$ that is defined by the functions $(Z(t), \hat{S}(t))$,

$$\Phi(x,t,\hbar) = \varphi\left(\frac{\Delta x}{\sqrt{\hbar}}, t, \hbar\right) \exp\left[\frac{i}{\hbar}(\hat{S}(t) + P(t)\Delta x)\right], \tag{44}$$

$$\hat{S} = S + \int_0^t \left[\frac{P(t)^2}{2m} + \frac{k}{2}X(t)^2 + lX(t) - \dot{X}(t)P(t) + \hat{g}V_0 - \hat{g}\frac{V_0}{2\gamma^2}\alpha^{(2)}_\Phi\right]. \tag{45}$$

The following estimates are valid for the functions $\Phi \in \hat{\mathcal{P}}^t_\hbar$ (44) in terms of Definition 2:

$$\Delta x = \hat{O}\left(\sqrt{\hbar}\right), \qquad \hat{\Delta p} = \hat{O}\left(\sqrt{\hbar}\right), \qquad -i\hbar\partial_t - \dot{\hat{S}}(t) + \dot{X}(t)\hat{p} - \dot{P}(t)\Delta x = \hat{O}(\hbar), \tag{46}$$

$$\Delta x = x - X(t), \qquad \Delta p = p - P(t), \qquad \hat{p} = -i\hbar\partial_x. \tag{47}$$

Let us expand the exponential in equation (41) in a Taylor series of $\Delta x = x - X(t)$, $\Delta y = y - X(t)$ and restrict ourselves to the terms of the order of not above four in Δx and Δy. In view of the estimates (46), (47) equation (41) takes the form

$$\begin{aligned}\Big\{\hat{L}_0 + \dot{\hat{S}} - \dot{X}(t)\hat{p} + \dot{P}(t)\Delta x + \dot{X}(t)P(t) + \frac{1}{m}P(t)\hat{\Delta p} + kX(t)\Delta x + l\Delta x \\ + \frac{\hat{g}V_0}{\gamma^2}\Delta x\alpha^{(1)}_\Phi + \hat{L}_1\Big\}\Phi = \hat{O}\left(\hbar^{5/2}\right),\end{aligned} \tag{48}$$

where

$$\hat{L}_0 = -i\hbar\partial_t - \dot{\hat{S}}(t) + \dot{X}(t)\hat{p} - \dot{P}(t)\Delta x + \frac{1}{2m}\hat{\Delta p}^2 + \frac{1}{2}\left(k - \frac{\hat{g}V_0}{\gamma^2}\right)\Delta x^2 = \hat{O}(\hbar), \tag{49}$$

$$L_1 = \frac{\hat{g}V_0}{8\gamma^4}\left(\Delta x^4 - 4\Delta x^3\alpha^{(1)}_\Phi + 6\Delta x^2\alpha^{(2)}_\Phi - 4\Delta x\alpha^{(3)}_\Phi + \alpha^{(4)}_\Phi\right) = \hat{O}\left(\hbar^2\right), \tag{50}$$

$$\alpha^{(k)}_\Phi(t,\hbar) = \frac{1}{\|\Phi\|^2}\int_{-\infty}^{\infty}(\Delta y)^k|\Phi(y,t)|^2dy, \quad k = 0,1,\ldots, \qquad \alpha^{(k)}_\Phi(t,\hbar) = O\left(\hbar^{k/2}\right). \tag{51}$$

Let us expand $\varphi(\xi,t,\hbar)$ in $\sqrt{\hbar}$ then

$$\Phi = \Phi^{(0)} + \sqrt{\hbar}\Phi^{(1)} + \hbar\Phi^{(2)} + \cdots, \qquad \Phi^{(k)} \in \mathcal{C}^t_\hbar, \tag{52}$$

$$\begin{aligned}\alpha^{(1)}_\Phi &= \alpha^{(1)}_{\Phi^{(0)}} + \sqrt{\hbar}\frac{2}{\|\Phi^{(0)}\|^2}\mathrm{Re}\langle\Phi^{(0)}|\Delta x|\Phi^{(1)}\rangle \\ &\quad + \hbar\frac{1}{\|\Phi^{(0)}\|^2}(\langle\Phi^{(1)}|\Delta x|\Phi^{(1)}\rangle + 2\mathrm{Re}\langle\Phi^{(0)}|\Delta x|\Phi^{(2)}\rangle).\end{aligned} \tag{53}$$

From (4) and (49)–(53) we have

$$\dot{\hat{S}} = 0, \qquad \dot{P}(t) = -kX(t) - l, \qquad \dot{X}(t) = \frac{1}{m}P(t), \tag{54}$$

$$\left(L_0 + \frac{\hat{g}V_0}{\gamma^2}\Delta x\alpha^{(1)}_{\Phi^{(0)}}\right)\Phi^{(0)} = 0, \tag{55}$$

$$\left(L_0 + \frac{\hat{g}V_0}{\gamma^2}\Delta x\alpha^{(1)}_{\Phi^{(0)}}\right)\Phi^{(1)} = -\frac{2}{\|\Phi^{(0)}\|^2}\frac{\hat{g}V_0}{\gamma^2}\Delta x\,\mathrm{Re}\langle\Phi^{(0)}|\Delta x|\Phi^{(1)}\rangle\Phi^{(0)}. \tag{56}$$

The function $\Phi^{(0)}$ is governed by (49), (55). It is defined as a linear Schrödinger equation with quadratic Hamiltonian that has the special solution (see, for example, [14, 15, 16]) in the form of Gaussian wave packet

$$\Phi^{(0)}_0 = N(t)\exp\left\{\frac{i}{\hbar}\left[a(t) + a_1(t)\Delta x + \frac{1}{2}f(t)\Delta x^2\right]\right\}, \qquad \mathrm{Im}\,f(t) > 0. \tag{57}$$

Here, the functions $a(t)$, $a_1(t)$, $f(t)$ are to be determined. With (49), (54), equation (55) takes the form

$$\Big\{-i\hbar\partial_t + \frac{1}{m}P(t)\hat{p} - \dot{P}(t)\Delta x + \frac{1}{2m}\hat{\Delta p}^2 + \frac{1}{2}\left(k - \frac{\hat{g}V_0}{\gamma^2}\right)\Delta x^2 + \frac{\hat{g}V_0}{\gamma^2}\Delta x \alpha^{(1)}_{\Phi_0^{(0)}}\Big\}\Phi_0^{(0)} = 0. \tag{58}$$

Note that for the Gaussian packet of general form we have

$$\alpha^{(1)}_{\Phi_0^{(0)}} = 0. \tag{59}$$

From (57)–(59) it follows that $a(t) = \text{const}$, $a_1(t) = P(t)$, $f(t) = \dot{C}(t)/C(t)$, $N(t) = C(t)^{-1/2}$, and (57) becomes

$$\Phi_0^{(0)} = \frac{1}{C^{1/2}} \exp\left\{\frac{i}{\hbar}\left[a + P(t)\Delta x + \frac{m}{2}\frac{\dot{C(t)}}{C(t)}\Delta x^2\right]\right\}. \tag{60}$$

With the initial conditions $C(0) = 1$, $B(0) = mb$, $\operatorname{Im} b < 0$, the function $C(t)$ can be found as follows:

$$1)\ \frac{1}{m}\left(k - \frac{\hat{g}V_0}{\gamma^2}\right) = \Omega^2 \geq 0, \qquad C(t) = \cos\Omega t + \frac{b}{\Omega}\sin\Omega t, \tag{61}$$

$$2)\ \frac{1}{m}\left(k - \frac{\hat{g}V_0}{\gamma^2}\right) = -\Omega^2 \leq 0, \qquad C(t) = \cosh\Omega t + \frac{b}{\Omega}\sinh\Omega t. \tag{62}$$

The variance of the coordinate x with respect to (60) will be

$$\alpha^{(2)}_{\Phi_0^{(0)}} = \frac{1}{\|\Phi_0^{(0)}\|^2}\int_{-\infty}^{\infty} \Delta x^2 |\Phi_0^{(0)}(x,t)| dx = \frac{\hbar |C(t)|^2}{2m \operatorname{Im}\left(\frac{\dot{C}}{C}\right)}. \tag{63}$$

It can be seen that for $\hat{g}V_0 < 0$ the variance $\alpha^{(2)}_{\Phi_0^{(0)}}(t,\hbar)$ is limited in t, i.e. $|\alpha^{(2)}_{\Phi_0^{(0)}}(t,\hbar)| \leq M$, $M = \text{const}$, while for $\hat{g}V_0 > 0$ it increases exponentially. In the limit of $\gamma \to 0$ and with $V_0 = (2\pi\gamma)^{-1/2}$, equation (4) becomes a nonlinear Schrödinger equation with the local nonlinearity, while in the case where $\hat{g}V_0 < 0$ ($\hat{g}V_0 > 0$) it corresponds to the condition of existence (nonexistence) of solitons.

Consider (60) as the vacuum solution of (58) regarded as the linear Schrödinger equation with quadratic Hamiltonian. Then the Fock basis of solutions of equation (58) yields a class of asymptotic solutions to the HTE. Due to the condition (59) the superposition principle is not fulfilled for these solutions. The last ones can be modified so that $\alpha^{(1)}_{\Phi_0^{(0)}} \neq 0$ and the superposition principle becomes valid.

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