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Geometric Formulation of Berezin Quantization

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In this paper we try to formulate the Berezin quantization on projective Hilbert space $\mathbb{P}(\mathcal{H})$ and use its geometric structure. It will be shown that the star product in Berezin quantization is equivalent to the Poisson bracket on $\mathbb{P}(\mathcal{H})$ and the Berezin method to construction a correspondence between a given classical theory and a given quantum theory is used to define a classical limit for geometric quantum mechanics.

1 Introduction

In Berezin quantization one defines from a representation of C^*-algebra of quantum observables the *covariant symbols*. These symbols are expectation values of the observables in terms of *coherent states*: the holomorphic functions on classical phase space M that is assumed to be a Kähler manifold.

Berezin [2] showed that the covariant symbols form a $*_\hbar$-algebra which in limit $\hbar \to 0$ leads to the Poisson algebra between the corresponding classical observables: The functions on the phase space M.

In this paper we will see that the Berezin $*_\hbar$-algebra is in fact a Poisson algebra which is induced by the Fubini–Study 2-form on space of coherent states. This space is defined as follows: coherent states span a dense subspace $\widetilde{\mathcal{H}}$ of Hilbert space $\mathcal{H}$. $\mathbb{P}(\widetilde{\mathcal{H}})$, which is denoted by $\mathcal{M}$, is a Kähler manifold with induced symplectic structure from $\mathbb{P}(\mathcal{H})$. Therefore the covariant symbols can be considered as functions on $\mathcal{M}$. It is shown [13] that there exists an embedding mapping between the classical phase space M and $\mathcal{M}$, by which to any point $z \in M$ is associated a point $Z \in \mathcal{M}$. With this construction to all of the quantum observables are associated their covariant symbols, which form a Poisson algebra on $\mathcal{M}$ and since the corresponding classical observables form a Poisson algebra on M, the Berezin quantization is a systematic procedure to relate these two Poisson algebras. Also the relation of Berezin quantization and geometric formulation of quantum mechanics will be evident as follows. The geometric quantum mechanics is a formulation of quantum mechanics in projective Hilbert spaces. With our construction one sees that the Berezin quantization is an equivalent formulation and in addition gives a prescription as classical limit for geometric quantum mechanics [9].

2 Geometry of projective Hilbert space

Let $\mathcal{H}$ be a Hilbert space and $\mathbb{P}(\mathcal{H})$ is its projective space by the canonical projection $\pi : \mathcal{H}\backslash 0 \to \mathbb{P}(\mathcal{H})$. Any point in $\mathbb{P}(\mathcal{H})$ is shown with $[\psi]$ corresponds to the one dimensional subspace $\mathbb{C}\psi$ in $\mathcal{H}$. $\mathbb{P}(\mathcal{H})$ is a Kähler manifold, the symplectic form is given by [12]

$$\Omega^\hbar_{[\psi]}(T_\psi\pi(\phi_1), T_\psi\pi(\phi_2)) = -2\hbar(\langle\phi_1, \phi_2\rangle), \tag{1}$$

where $\phi \in (\mathbb{C}\psi)^{\perp}$ and $T_{\psi}\pi(\phi)$ is tangential space of $\mathbb{P}(\mathcal{H})$ in point $[\psi]$, which is isomorphic to $\mathcal{H}\backslash\mathbb{C}\psi$,

$$T_{\psi}\pi : \mathcal{H} \to T_{[\psi]}\mathbb{P}(\mathcal{H}) \simeq \mathcal{H}\backslash\mathbb{C}\psi. \tag{2}$$

and defined by

$$(T_{\psi}\pi)(\phi) = \frac{d}{dt}\pi(\psi + t\phi) \mid_{t=0} .$$

2.1 Vector fields on $\mathbb{P}(\mathcal{H})$

Let (M,ω) be a symplectic manifold. The vector field A is called *Hamiltonian* if there exists a smooth function f on M such that

$$i_A\omega = df, \tag{3}$$

where i_A is interior derivative of Ω with respect to A.

The quantum mechanical observables are self adjoint operators on Hilbert space and one can consider the expectation values of these observables as function on projective Hilbert space; in fact the expectation value of H_A is defined by

$$\langle H_A\rangle_{\psi} = \frac{\langle\psi, H_A\psi\rangle}{\langle\psi,\psi\rangle}. \tag{4}$$

By the following theorem the relation between the operators on Hilbert space and the associated Hamiltonian vector field will be evident.

Theorem 1. *Let A be a Hamiltonian vector field on $\mathbb{P}(\mathcal{H})$ and H_A the corresponding Hamiltonian operator on $\mathcal{H}$. Then the Schrödinger equation $H_A\psi(t) = i\hbar d\psi/dt$ is equivalent to the equation of motion that induced by A on $\mathbb{P}(\mathcal{H})$, such that*

$$A[\psi] = \frac{1}{i\hbar}\frac{H_A\psi}{\|\psi\|}, \tag{5}$$

where A is given in local coordinates Z on $\mathbb{P}(\mathcal{H})$ with respect to Fubini-study form as

$$A = -i\sum_{n,p}\Omega^{k,np}\left(\frac{\partial\langle H_A\rangle}{\partial\bar{Z}^k_p}\frac{\partial}{\partial Z^k_n} - \frac{\partial\langle H_A\rangle}{\partial Z^k_p}\frac{\partial}{\partial\bar{Z}^k_n}\right). \tag{6}$$

Proof. See [3, 15]. ■

As a consequence one can say that the Schrödinger equation is nothing but the classical Hamilton equations. Then it is natural to expect that there exists a Poisson structure on $\mathbb{P}(\mathcal{H})$. With the following proposition will be seen that the symplectic form on $\mathbb{P}(\mathcal{H})$ endows it with Poisson algebra. For a symplectic manifold with form Ω we have:

$$\{f,g\} = \Omega(X_f, X_g), \tag{7}$$

where X_f and X_g are Hamiltonian vector fields of f and g respectively.

Proposition 1. *Let $A, B : \mathbb{P}(\mathcal{H}) \to T\mathbb{P}(\mathcal{H})$ are two Hamiltonian vector fields corresponding to the functions $\langle H_A\rangle$ and $\langle H_B\rangle$ on $\mathbb{P}(\mathcal{H})$ respectively. Then*

$$\{\langle H_A\rangle, \langle H_B\rangle\} = \langle\frac{1}{i\hbar}[H_A, H_B]\rangle, \tag{8}$$

where the Poisson bracket is defined by (1) and the relation

$$\{\langle H_A\rangle, \langle H_B\rangle\} = \Omega_{FS}(A,B).$$

Proof. Direct calculation [12, 15]. ■

It must be pointed out that the Poisson structure is defined on quantum phase space $\mathbb{P}(\mathcal{H})$ rather than classical phase space M.

It is well known that the $\mathbb{P}(\mathcal{H})$ has a natural metric, called Fubini–Study metric g, by which the transition probability is defined [3, 9, 12]. Then the vector field A on $\mathbb{P}(\mathcal{H})$ is Hamiltonian if and only if $\mathcal{L}_A g = 0$, where $\mathcal{L}_A$ is Lie derivative along A and $A = X_{\langle H_A\rangle}$ is defined by (3). Therefore the Hamiltonian flow of the functions $\langle H_A\rangle$ preserves the geometric structures carried by $\mathbb{P}(\mathcal{H})$ and then the quantum mechanical observables generate the structural symmetries of $\mathbb{P}(\mathcal{H})$ [4].

2.2 The coherent states manifold

The generalized coherent states are elements of a G-orbit, which are generated by action of the Lie group G on a dominant weight vector ϕ_0 in the separated Hilbert space $\mathcal{H}$. This orbit $\widetilde{\mathcal{H}}$ is dense subspace of $\mathcal{H}$ [1]. If U_g is a unitary representation of $g \in G$ Then the projective space $\mathbb{P}(\widetilde{\mathcal{H}}) \equiv \mathcal{M}$ is also a dense subspace of $\mathbb{P}(\mathcal{H})$. $\mathcal{M}$ is Kählerian if G is a semi-simple group [11]. The manifold of coherent states is given by

$$\mathcal{M} = \{[U_g\psi_0] \mid g \in G\}, \tag{9}$$

where $[U_g]$ is the projective representation of G induced by U.

Let $\mathcal{K}$ denote the maximal stabilisator of G. Then there exists an isomorphism between $\mathcal{M}$ and $G/\mathcal{K}$.

With this construction there exist an embedding $\iota : \mathcal{M} \to \mathbb{P}(\mathcal{H})$ and the symplectic and other geometrical structures of projective Hilbert space are induced in $\mathcal{M}$:

$$\Omega = \iota^*\Omega_{FS} = \Omega_{FS}|_M. \tag{10}$$

2.3 The embedding of classical phase space in $\mathcal{M}$

Let (M, ω) be a Kähler manifold as classical phase space. We define the weighted Bergman space as

$$\mathcal{H}_\hbar = \left\{ f| \int |f(z)|^2 e^{-\frac{1}{\hbar}\Psi(z,\bar{z})} d\nu(z,\bar{z}) = \|f\|_\hbar^2 < \infty \right\}. \tag{11}$$

As a subspace of $L^2\left(M, e^{-\frac{1}{\hbar}\Psi}\right)$, $\mathcal{H}_\hbar$ is a Hilbert space. In fact $\mathcal{H}_\hbar$ is the space of analytic quadratic integrable functions on Kähler manifold M with measure

$$d\mu(z,\bar{z}) = e^{-\frac{1}{\hbar}\Psi(z,\bar{z})} d\nu(z,\bar{z}). \tag{12}$$

In this space the Berezin coherent states $\Phi^\hbar_{\bar{\zeta}}$ form a overcomplete set. According to definition of inner product in $\mathcal{H}_\hbar$ we have

$$\Phi^\hbar_{\bar{\zeta}}(z) = \langle \Phi^\hbar_{\bar{z}}, \Phi^\hbar_{\bar{\zeta}}\rangle_\hbar =: K_\hbar(\bar{\zeta}, z), \tag{13}$$

where $K_\hbar(\bar{\zeta}, z)$ is the Bergman kernel, which is defined uniquely for any manifold and has the reproducing property

$$f(\zeta) = \langle \Phi^\hbar_{\bar{\zeta}}, f\rangle_\hbar. \tag{14}$$

For a symmetric space the Berezin coherent states are the same as the generalized coherent states [14]. Therefore to any point of Kähler manifold M is associated a coherent state in $\mathcal{H}_\hbar$ as a kerned Hilbert space. Hence there exists a holomorphic embedding $\iota_\hbar : M \to \mathcal{M}_\hbar$, where $\mathcal{M}_\hbar$

is the projective space of $\mathcal{H}_\hbar$. This association is called the coherent states quantization [1, 13]. Two important properties of this embedding are that it is one-one and global differentiable. Then the pull-back of $\iota^*\Omega_{FS}$ of Fubini–Study form of $\mathcal{M}_\hbar$, induced from $\mathcal{M}$, is again a symplectic form. If the coherent states are generated from the representation of a Lie group G, then (M,ω) is a homogeneous symplectic manifold.

3 The Berezin quantization on the coherent states manifold

Berezin quantization [2, 6] on an arbitrary Kähler manifold is defined by the $*_\hbar$-algebra out of covariant symbols, which are the expectation values of quantum observables (self adjoint bounded operators) in terms of coherent states $\Phi^\hbar_{\bar\zeta}$

$$\widetilde{AB} = \widetilde{A} *_\hbar \widetilde{B}(z) = \int_M \widetilde{A}(\bar\zeta, z)\widetilde{B}(\bar z, \zeta)\frac{|K_\hbar(\bar\zeta, z)|^2}{K_\hbar(\bar z, z)} e^{-\frac{1}{\hbar}\Psi(\bar\zeta, z)}\, d\nu(\bar\zeta, \zeta), \tag{15}$$

where $\widetilde{A}$ is Berezin covariant symbol defined by

$$\widetilde{A}(\bar\zeta, z) = \frac{\langle K_\hbar(\bar z,\cdot), AK_\hbar(\bar\zeta,\cdot)\rangle_\hbar}{K_\hbar(\bar\zeta, z)} = \frac{\langle \Phi^\hbar_{\bar z}, A\Phi^\hbar_{\bar\zeta}\rangle_\hbar}{\langle \Phi^\hbar_{\bar z}, \Phi^\hbar_{\bar\zeta}\rangle_\hbar}. \tag{16}$$

$K_\hbar(\bar\zeta, z)$ is the Bergman kernel and $\Psi(\bar\zeta, z)$ is the Kähler function. Berezin has considered the covariant symbols as bounded functions on classical phase space (M,ω), to be a Kähler manifold, which form the $*_\hbar$-algebra $\mathcal{A}_\hbar$. The classical limit $\hbar \to 0$ results from

$$\left(\widetilde{A} *_\hbar \widetilde{B}\right)(z) = a(z)b(z) + \mathcal{O}(\hbar), \tag{17}$$

$$\frac{1}{\hbar}\left(\widetilde{A} *_\hbar \widetilde{B} - \widetilde{B} *_\hbar \widetilde{A}\right)(z) = -i\{a,b\}(z) + \mathcal{O}(\hbar). \tag{18}$$

where a, b are the $\hbar \to 0$ limits of $\widetilde{A}$, $\widetilde{B}$ respectively.

By construction in Section 2 we can also consider the covariant symbols as functions on projective Hilbert space $\mathcal{M}_\hbar$. Therefore these functions form a Poisson algebra via the induced Fubini-Study form on $\mathcal{M}_\hbar$. What we must show is that both these algebras, i.e. Poisson algebra and $*_\hbar$ algebra, are the same.

From Proposition 1 one sees clearly that for two covariant symbols $\widetilde{A}$, $\widetilde{B}$, as expectation values of the operators A, B, in terms of coherent states, we have

$$\frac{1}{i\hbar}\widetilde{[A,B]} = \{\widetilde{A}, \widetilde{B}\}_{\iota^*\Omega_{FS}}. \tag{19}$$

On other hand from equation (15) it can be easily shown

$$\frac{1}{i\hbar}\widetilde{[A,B]} = \frac{1}{i\hbar}\left(\widetilde{A} *_\hbar \widetilde{B} - \widetilde{B} *_\hbar \widetilde{A}\right). \tag{20}$$

The lhs of equations (19) and (20) are identical, so we have

$$\frac{1}{i\hbar}\left(\widetilde{A} *_\hbar \widetilde{B} - \widetilde{B} *_\hbar \widetilde{A}\right) = \left\{\widetilde{A}, \widetilde{B}\right\}_{\iota^*\Omega_{FS}}. \tag{21}$$

Hence: *The $*_\hbar$-algebra correspond to Poisson algebra on $\mathcal{M}_\hbar$.*

We emphasise again that this Poisson structure is defined on quantum phase space and preserves all of the quantum mechanical properties of the system.

The classical limit in Berezin quantization is defined now by

$$\lim_{\hbar\to 0}\left\{\widetilde{A},\widetilde{B}\right\}_{\iota^*\Omega_{FS}}(Z)=\left\{\varphi(\widetilde{A}),\varphi(\widetilde{B})\right\}(z),\qquad z\in M, Z\in\mathcal{M}_\hbar, \tag{22}$$

where φ is defined as the quantum to classical observable map:

$$\lim_{\hbar\to 0}\widetilde{A}=\varphi(\widetilde{A}). \tag{23}$$

Dynamics is also defined in Berezin quantization as follows:

The Heisenberg equation of motion for the observable A is $\frac{dA}{dt}=\frac{1}{i\hbar}[A,H]$. This equation on $\mathcal{M}_\hbar$ has the following form

$$\frac{d\widetilde{A}(Z)}{dt}=\left\{\widetilde{A}(Z),\widetilde{H}(Z)\right\}_{\iota^*\Omega_{FS}},\qquad Z\in\mathcal{M}. \tag{24}$$

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