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Generalized Canonical Commutation Relations: Representations and Stability of Universal Enveloping C^*-Algebra

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We consider the deformation of canonical commutation relations in the class of Wick algebras. The irreducible representations of GCCR are classified. We study the universal bounded representation of GCCR and compute the K-theory for the twisted canonical commutation relations.

In this paper we study the deformation of CCR generalising both twisted CCR of W. Pusz and S.L. Woronowicz and some type of q_{ij}-CCR of M. Bozejko and R. Speicher (see [3, 1]). Namely, let us consider a $*$-algebra generated by elements $\{a_i, a_i^*,\ i = 1, \dots, d\}$ satisfying the following relations (GCCR)

$$\begin{aligned}
&a_i^* a_i = 1 + \alpha_i a_i a_i^* - \sum_{j<i:\ k_j \ge i} (1-\alpha_j) a_j a_j^*,\\
&a_i^* a_j = \lambda_{ij} \alpha_i a_j a_i^*, \qquad a_j a_i = \lambda_{ij} \alpha_i a_i a_j, \qquad i<j, \quad k_i \ge j,\\
&a_i^* a_j = \lambda_{ij} a_j a_i^*, \qquad a_j a_i = \lambda_{ij} a_i a_j, \qquad i<j, \quad k_i < j,\\
&0 < \alpha_i < 1, \qquad |\lambda_{ij}| = 1, \qquad i,j = 1,\dots,d, \quad i \ne j,
\end{aligned} \tag{1}$$

where the vector $\vec{k} = (k_1, k_2, \dots, k_{d-1})$ has the property that $d \ge k_i \ge i$ and if $j < i$ and $i \le k_j$ then $k_i \le k_j$.

Example 1. For $\vec{k} = (d, \dots, d)$, $\alpha_i = \mu^2$, $i = 1, \dots, d$ and $\lambda_{ij} = 1$, $i \ne j$ we have a well-known twisted CCR:

$$\begin{aligned}
&a_i^* a_i = 1 + \mu^2 a_i a_i^* - \left(1-\mu^2\right) \sum_{j<i} a_j a_j^*,\\
&a_i^* a_j = \mu a_j a_i^*, \qquad a_j a_i = \mu a_i a_j, \qquad i<j.
\end{aligned}$$

Example 2. If we put $\vec{k} = (1, 2, \dots, d-1)$ we get

$$a_i^* a_i = 1 + \alpha_i a_i a_i^*, \qquad a_i^* a_j = \lambda_{ij} a_j a_i^*, \qquad a_j a_i = \lambda a_j a_i, \qquad i<j,$$

i.e. the so-called generalised "quon" commutation relations, which form a special type of q_{ij}-CCR.

In the Section 1 we study the C^*-algebra A constructed by the bounded representations of these relations and show that it is isomorphic to the $C^*(s_i, s_i^*)$ where partial isometries s_i, s_i^* satisfy the relations

$$s_i^* s_i = 1 - \sum_{j<i:\ k_j \geq i} s_j s_j^*, \qquad s_i^* s_j = 0, \qquad s_j s_i = 0, \qquad i<j, \qquad k_i \geq j,$$
$$s_i^* s_j = \lambda_{ij} s_j s_i^*, \qquad s_j s_i = \lambda_{ij} s_i s_j, \qquad i<j, \qquad k_i < j.$$

As a corollary of this stability result we have that $K_0(A_\mu) = \mathbb{Z}$ and $K_1(A_\mu) = \{0\}$, where A_μ is the C^*-algebra associated with TCCR. We also prove that Fock representation of A is faithful.

In the Section 2 we study the unbounded representations of (GCCR) for several particular choices of parameters.

1 The universal bounded representations

The bounded representations of (1) were studied in [2]. Let π be irreducible bounded representation of GCCR. Denote $\pi(a_i)$ by A_i and consider the polar decomposition $A_i^* = U_i^* C_i$, $i = 1, \ldots, d$. Let us fix some subset $\Phi \subset \{1, \ldots, d\}$. Put $\Theta = \cup_{j\in\Phi}[j, k_j]$. Then the irreducible bounded representation of GCCR corresponding to Φ has the following form

$$C_i = U_i = 0, \qquad i \in \Theta,$$
$$C_i^2 = \bigotimes_{j<i, j\notin\Theta} d_{ij} \otimes D_i^2 \bigotimes_{j>i, j\notin\Theta} 1, \qquad i \notin \Phi,$$
$$U_i^* = \bigotimes_{j<i, j\notin\Theta} U_{ij} \otimes S \bigotimes_{j>i, j\notin\Theta} 1, \qquad i \notin \Phi,$$
$$C_i^2 = \frac{1}{1-\alpha_i} \bigotimes_{j<i, j\notin\Theta} d_{ij} \otimes \bigotimes_{j\geq i, j\notin\Theta} 1, \qquad i \in \Phi,$$
$$U_i^* = \bigotimes_{j<i, j\notin\Theta} U_{ij} \otimes \bigotimes_{j>i, j\notin\Theta} U_{ij} \otimes \widehat{U}_i, \qquad i \in \Phi,$$

where $d_{ij}\colon l_2(\mathbb{N}) \to l_2(\mathbb{N})$

$$d_{ij} e_n = \alpha_j^{n-1} e_n, \qquad k_j \geq i, \qquad d_{ij} = 1, \qquad k_j < i,$$

$D_i^2\colon l_2(\mathbb{N}) \to l_2(\mathbb{N})$, $D_i^2 e_n = \frac{1-\alpha_i^{n-1}}{1-\alpha_i} e_n$, $U_{ij} e_n = \lambda_{ij}^{n-1} e_n$ and the family of unitary operators $\widehat{U}_i$, $i \in \Phi$ is irreducible and satisfies the relations

$$\widehat{U}_i \widehat{U}_j = \lambda_{ij} \widehat{U}_j \widehat{U}_i.$$

It can be easily seen that for any bounded representation we have the norm bound

$$\|\pi(a_i^* a_i)\| \leq \frac{1}{1-\alpha_i}.$$

Hence one can construct the universal bounded representation of GCCR, i.e. the C^*-algebra $A_{\alpha,\lambda}$ generated by a_i, a_i^* with norm:

$$\|X\| = \sup_\pi \|\pi(X)\|,$$

where X is any element of a $*$-algebra generated by GCCR, and sup is taken over all irreducible representations of GCCR.

Theorem 1. *The C^*-algebra $A_{\alpha,\lambda}$ is isomorphic to $A_{0,\lambda}$ for any choice of parameters α_i, $i = 1,\dots,d$, $0 < \alpha_i < 1$ where $A_{0,\lambda}$ is generated by partial isometries s_i, $i = 1,\dots,d$ satisfying the relations*

$$s_i^* s_i = 1 - \sum_{j<i:\ k_j \ge i} s_j s_j^*, \qquad s_i^* s_j = 0, \qquad s_j s_i = 0, \qquad i < j, \qquad k_i \ge j,$$

$$s_i^* s_j = \lambda_{ij} s_j s_i^*, \qquad s_j s_i = \lambda_{ij} s_i s_j, \qquad i < j, \qquad k_i < j.$$

For the particular case of TCCR we have the C^*-algebra A_0 generated by the relations

$$s_i^* s_j = \delta_{ij} \left(1 - \sum_{k<i} s_i s_i^* \right).$$

In the following theorem we suppose that the coefficients $\lambda_{ij} = e^{2\pi\theta_{ij}}$ have the additional property that the family $\{1, \theta_{ij}\}$ is linearly independent over $\mathbb{Q}$.

Theorem 2. *The Fock representation of $A_{0,\lambda}$ is faithful.*

For example for C^*-algebra generated by TCCR we have the following faithful realization

$$s_i = \bigotimes_{j<i} (1 - SS^*) \otimes S \otimes \bigotimes_{j>i} 1, \qquad i = 1,\dots,d.$$

Using this realization we compute the K-groups of A_0.

Theorem 3. $K_0(A_\mu) \simeq \mathbb{Z}$ *and* $K_1(A_\mu) = \{0\}$.

Proof. As it was noted above that $A_\mu \simeq C^*(s_i, s_i^*)$, where

$$s_i = \bigotimes_{j<i} (1 - ss^*) \otimes s \otimes \bigotimes_{j>i} 1, \qquad i = 1,\dots,d.$$

Let us consider the case $d = 2$. Let $\widetilde{\mathcal{T}_0}$ be the ideal generated by the element $(1 - ss^*) \otimes (1 - s)$. It is easy to see that $\widetilde{\mathcal{T}_0} \simeq \mathcal{K} \otimes \mathcal{T}_0$, where $\mathcal{T}_0$ is an ideal in the Toeplitz algebra $\mathcal{T}$ generated by the element $1 - s$. It is known fact that $K_i(\mathcal{T}_0) = \{0\}$. Further, $A_\mu/\widetilde{\mathcal{T}_0} \simeq \mathcal{T}$, i.e. we have the following short exact sequence

$$0 \longrightarrow \widetilde{\mathcal{T}_0} \longrightarrow A_\mu \longrightarrow \mathcal{T} \longrightarrow 0.$$

Since $K_0(\mathcal{T}) \simeq \mathbb{Z}$ and $K_1(\mathcal{T}) = \{0\}$, the corresponding six-term exact sequence becomes

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(A_\mu) & \longrightarrow & \mathbb{Z} \\ \big\downarrow & & & & \big\downarrow \\ 0 & \longleftarrow & K_1(A_\mu) & \longleftarrow & 0 \end{array}$$

In the general case we consider the ideal $\widehat{\mathcal{T}_0}$ generated by the element $\bigotimes_{i=1}^{d-1} (1 - ss^*) \otimes (1 - s)$. Then

$$\widehat{\mathcal{T}_0} \simeq \bigotimes_{i=1}^{d-1} \mathcal{K} \otimes \mathcal{T}_0 \simeq \mathcal{K} \otimes \mathcal{T}_0$$

and $A_0(d)/\widehat{\mathcal{T}_0} \simeq A_0(d-1)$. Applying again the six-term sequence corresponding to the

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{T}_0 \longrightarrow A_0(d) \longrightarrow A_0(d-1) \longrightarrow 0$$

and induction on d, we get $K_0(A_0(d)) \simeq \mathbb{Z}$ and $K_1(A_0(d)) = \{0\}$. ∎

2 Representations of GCCR

In this section we restrict ourselves by the case $\alpha_i = \mu^2$ for any $i = 1, \dots, d$. To give the classification of irreducible representations of GCCR we need to introduce some notations. Let

$$\Phi = \{1 \le i_1 < i_2 < \cdots < i_m \le d \mid i_j > k_{i_{j-1}}\}.$$

Consider the function $l\colon \Phi \to \mathbb{N}$ such that for any $j \in \Phi$ we have $j \le l(j) \le k_j$. Construct the set Ψ:

$$\Theta = \cup_{j\in\Phi}([j+1, l(j)] \cap \mathbb{Z}),$$
$$\Psi := \{l(j)+1 \mid l(j)+1 \le k_j,\ j \in \Phi\}.$$

Let $F(j) := k_j + 1$, $j = 1, \dots, d-1$. For any $s \in \Psi$ denote by Ψ_s the following set:

$$\Psi_s = \{F^n(s),\ n \in \mathbb{Z}_+\} \cap [s, k_{m(s-1)}],$$

where $l(m(s-1)) = s-1$ for any $s \in \Psi$. Put also $M = \{1 \le j_1 < \cdots < j_t \le d\}$, such that $j_i \notin \cup_{l\in\Phi[l,k_l]}$ and $j_i > k_{j_{i-1}}$, $i = 1, \dots, t$. Finally, let

$$F = \{1, \dots, d\} \setminus \big(\cup_{i\in\Phi}[i, k_i] \cup \cup_{j\in M}[j, k_j]\big).$$

For any $j \in \Psi_s$, $s \in \Psi$ fix some $z_{js} > 0$ and put $\tau_{js} := \big(\mu^2 z_{js}, z_{js}\big]$. Fix any $x_{js} \in \tau_{js}$ and construct the function

$$g(x, x_{js}) = -\left(1-\mu^2\right) x_{js} + \mu^2 x.$$

For any $i \in M$ fix $y_i > \frac{1}{1-\mu^2}$ and set $\tau_i := (1 + \mu^2 y_i, y_i]$. For any $x_i \in \tau_i$, $i \in M$ consider the function

$$f(x, x_i) = 1 - \left(1-\mu^2\right) x_i + \mu^2 x.$$

As in the bounded case we give the description of representations of GCCR using the polar decompositions $\pi(a_i^*) = U_i C_i$.

Theorem 4. *The irreducible representations of GCCR have (up to the unitary equivalence) the following form*

$$C_i^2 = \bigotimes_{j<i, j\notin\Theta} d_{ij} \otimes D_i^2 \otimes \bigotimes_{j>i, j\notin\Theta} 1, \qquad i \notin \Phi\cup\Theta,$$
$$U_i^* = \bigotimes_{j<i, j\notin\Theta} U_{ij} \otimes U_i \otimes \bigotimes_{j>i, j\notin\Theta} U_{ij}, \qquad i \notin \Phi\cup\Theta,$$
$$C_i^2 = \frac{1}{1-\mu^2} \bigotimes_{j<i, j\notin\Theta} d_{ij} \otimes D_i^2 \otimes \bigotimes_{j>i, j\notin\Theta} 1, \qquad i \in \Phi,$$
$$U_i^* = \bigotimes_{j<i, j\notin\Theta} U_{ij} \otimes U_i \otimes \bigotimes_{j>i, j\notin\Theta} U_{ij} \otimes \widehat{U}_i, \qquad i \in \Phi,$$
$$C_i^2 = 0, \qquad U_i = 0, \qquad i \in \Theta,$$

where $\{\widehat{U}_i,\ i \in \Phi\}$ form the irreducible representation of higher-dimensional non-commutative torus, i.e.

$$\widehat{U}_i \widehat{U}_j = \lambda_{ij} \widehat{U}_j \widehat{U}_i, \qquad i \ne j$$

and

$$
\begin{aligned}
D_i^2 &= D\left(\mu^2, x_{is}\right) : l_2(\mathbb{Z}) \to l_2(\mathbb{Z}), \qquad x_{is} \in \left(\mu^2 z_{is}, z_{is}\right], \qquad i \in \Psi_s, \qquad s \in \Psi,\\
&\quad D\left(\mu^2, x_{is}\right) e_n = \mu^{2n} x_{is} e_n, \qquad n \in \mathbb{Z},\\
D_i^2 &= d\left(g^{-n}(0, x_{js})\right) : l_2(\mathbb{Z}_-) \to l_2(\mathbb{Z}_-), \qquad i \in [j+1, k(j)], \qquad j \in \Psi_s,\\
&\quad d\left(g^{-n}(0, x_{js})\right) e_{-n} = g^{-n}(0, x_{js}) e_{-n}, \qquad n \in \mathbb{Z}_+,\\
D_i^2 &= D\left(f^n(x_i)\right) : l_2(\mathbb{Z}) \to l_2(\mathbb{Z}), \qquad x_i \in \left(1 + \mu^2 y_i, y_i\right], \qquad i \in M,\\
&\quad D\left(f^n(x_i)\right) e_n = f^n(x_i) e_n, \qquad n \in \mathbb{Z},\\
D_i^2 &= d\left(f^{-n}(0, x_j)\right) : l_2(\mathbb{Z}_-) \to l_2(\mathbb{Z}_-), \qquad i \in [j+1, k_j], \qquad j \in M,\\
&\quad d\left(f^{-n}(o, x_i)\right) e_{-n} = f^{-n}(0, x_i) e_{-n}, \qquad n \in \mathbb{Z}_+,\\
D_i^2 &= d\left(f^n(0)\right) : l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+), \qquad i \in F,\\
&\quad d\left(f^n(0)\right) e_n = f^n(0) e_n, \qquad n \in \mathbb{Z}_+
\end{aligned}
$$

and

$$
\begin{aligned}
U_i^* &= U \colon l_2(\mathbb{Z}) \to l_2(\mathbb{Z}), \qquad i \in \Psi_s, \qquad s \in \Psi, \qquad U e_n = e_{n+1},\\
U_i^* &= \widehat{S} \colon l_2(\mathbb{Z}_-) \to l_2(\mathbb{Z}_-), \qquad i \in [j+1, k(j)], \qquad j \in \Psi_s,\\
&\quad \widehat{S} e_{-n} = e_{-n+1}, \qquad n \in \mathbb{N}, \qquad \widehat{S} e_0 = 0,\\
U_i^* &= U, \qquad i \in M,\\
U_i^* &= \widehat{S}, \qquad i \in [j+1, k_j], \qquad j \in M,\\
U_i^* &= S \colon l_2(\mathbb{N}) \to l_2(\mathbb{N}), \qquad i \in F, \qquad S e_n = e_{n+1}.
\end{aligned}
$$

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