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Lie, Partially Invariant, and Nonclassical Submodels of Euler Equations

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The Euler equations describing motion of an incompressible ideal fluid are investigated with symmetry point of view. We review some results on Lie, partially invariant, and nonclassical submodels of these equations.

1 Introduction

Hydrodynamics partial differential equations are traditional objects of investigation by means of methods of group analysis [1]. It is well known [2, 3] that the maximal Lie invariance algebra of the Euler equations (EEs)

$$\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p = \vec{0}, \qquad \operatorname{div}\vec{u} = 0, \tag{1}$$

which describe flows of an ideal incompressible fluid, is the infinite dimensional algebra $A(E)$ generated by the following basis elements:

$$\begin{aligned}
&\partial_t, \quad J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} \quad (a<b),\\
&D^t = t\partial_t - u^a\partial_{u^a} - 2p\partial_p, \quad D^x = x_a\partial_a + u^a\partial_{u^a} + 2p\partial_p,\\
&R(\vec{m}) = R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p,\\
&Z(\chi) = Z(\chi(t)) = \chi(t)\partial_p.
\end{aligned} \tag{2}$$

Such anomalously wide Lie invariance is typical for hydrodynamics equations of incompressible fluids, which are written in the Euler coordinates.

In the following $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity of the fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian, $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbb{R})$). The fluid density is set equal to unity. Summation over repeated indices is implied, and we have $a,b = 1,2,3$. Subscripts of functions denote differentiation with respect to the corresponding variables.

2 Lie invariant solutions of Euler equations

A number of Lie submodels of (1) have been already constructed. For example, in [4, 5, 6, 7] EEs are reduced to partial differential equations in two and three independent variables by means of using the Lie algorithm.

Using well-known Lie symmetry group of EEs, we describe all its possible (inequivalent) Lie submodels. Namely, we find complete sets of inequivalent one-, two-, and three-dimensional subalgebras of $A(E)$. Then, we construct the corresponding ansatzes of codimension one, two, and three as well as reduced systems of partial differential equations in three and two independent variables and reduced systems of ordinary differential equations. Lie symmetry properties of the reduced systems of partial differential equations are investigated. There exists a number of reduced systems admitting Lie symmetries which are not induced by Lie symmetries of the initial Euler equations. (Existence of such symmetries was firstly proved by L.V. Kapitanskiy [8, 9] just for the axially symmetric Euler equations.) The reduced systems of ordinary differential equations are integrated or for them partial exact solutions are found. As a result, new large classes of exact solutions of EEs, which contain, in particular, arbitrary functions, are constructed. Numbers of investigated objects are the following ones:

5 families of one-dimensional inequivalent subalgebras
5 families of ansatzes of codimension one (all the families of subalgebras can be used to reduce the EEs by the standard method)
4 classes of reduced systems (two classes of reduced systems can be united)
2 classes of reduced systems that have non-induced Lie symmetries

16 families of two-dimensional inequivalent subalgebras
14 families of ansatzes of codimension two (14 subalgebras can be used to reduce EEs by the standard method)
11 classes of reduced systems (there exist 3 pairs of classes of reduced systems, which can be united)
2 classes of reduced systems are completely integrated
1 reduced system is linearized on a subset of solutions

45 families of three-dimensional inequivalent subalgebras
21 families of ansatzes of codimension three (only 21 families of subalgebras can be used to reduce EEs by the standard method)
10 classes of reduced systems solutions of which are not solutions of completely integrated reduced systems with two independent variables

Now we consider two stationary Lie submodels of codimension 3, which do not have analogs in the case of viscous fluids as their construction essentially bases on specific invariance of EEs with respect the time dilations generated by the operator D^t. Moreover, integrating of these nonlinear submodels can be reduced to solving of second order linear ODEs. Below we give the corresponding subalgebras, ansatzes, reduced systems, and their solutions.

1. $\langle \partial_t,\ J_{12}+\alpha_1 D^t,\ R(0,0,1)+\alpha_2 D^t\rangle$, where $(\alpha_1,\alpha_2)\neq(0,0)$;

$$u^1=(x_1\varphi^1-x_2\varphi^2)e^{\zeta},\quad u^2=(x_2\varphi^1+x_1\varphi^2)e^{\zeta},\quad u^3=\varphi^3e^{\zeta},\quad p=he^{2\zeta},$$

where $\zeta=-\alpha_2x_3-\alpha_1\arctan x_2/x_1$, $\omega=(x_1^2+x_2^2)^{1/2}$, and new unknown functions $\varphi^a=\varphi^a(\omega)$ and $h=h(\omega)$ satisfy the reduced system

$$\begin{aligned}
&\omega\varphi^1\varphi^1_\omega-(\alpha_1\varphi^2+\alpha_2\varphi^3)\varphi^1+(\varphi^1)^2-(\varphi^2)^2+\omega^{-1}h_\omega=0,\\
&\omega\varphi^1\varphi^2_\omega-(\alpha_1\varphi^2+\alpha_2\varphi^3)\varphi^2+2\varphi^1\varphi^2-2\alpha_1\omega^{-2}h=0,\\
&\omega\varphi^1\varphi^3_\omega-(\alpha_1\varphi^2+\alpha_2\varphi^3)\varphi^3-2\alpha_2h=0,\\
&\omega\varphi^1_\omega+2\varphi^1-(\alpha_1\varphi^2+\alpha_2\varphi^3)=0.
\end{aligned}\tag{3}$$

If $\varphi^1=0$, then $h=\varphi^2=\alpha_2\varphi^3=0$ and we obtain a trivial solution of EEs. Let $\varphi^1\neq0$. It follows from system (3) that

$$\varphi^2=\frac{\alpha_1(\omega\varphi^1_\omega+2\varphi^1)+\alpha_2\beta\omega^2\varphi^1}{\alpha_2^2\omega^2+\alpha_1^2},\qquad \varphi^3=\frac{\alpha_2(\omega\varphi^1_\omega+2\varphi^1)-\alpha_1\beta\omega^2\varphi^1}{\alpha_2^2\omega^2+\alpha_1^2},$$

$$h=\frac{\omega^2}{2}\frac{\omega^2\varphi^1\varphi^1_{\omega\omega}-(\omega\varphi^1_\omega)^2-4(\varphi^1)^2}{\alpha_2^2\omega^2+\alpha_1^2}+\frac{\omega^2}{2}\varphi^1\frac{(\alpha_1^2-\alpha_2^2\omega^2)\omega\varphi^1_\omega+2\alpha_1\varphi^1(\alpha_2\beta\omega^2+2\alpha_1)}{(\alpha_2^2\omega^2+\alpha_1^2)^2},$$

where $\varphi^1=\omega^{-2}(\alpha_2^2\omega^2+\alpha_1^2)^{1/2}\psi(\omega)$, ψ is an arbitrary solution of the second order linear ODE

$$\psi_{\omega\omega}+\frac{1}{\omega}\psi_\omega+\left(\alpha_2^2\frac{\alpha_1^2-\alpha_2^2\omega^2}{(\alpha_2^2\omega^2+\alpha_1^2)^2}+\left(\beta+\frac{\alpha_1\alpha_2}{\alpha_2^2\omega^2+\alpha_1^2}\right)^2+(\alpha_2^2\omega^2+\alpha_1^2)(\omega^{-2}+\gamma)\right)\psi=0,$$

β and γ are arbitrary constants. For some values of parameters the general solution of the last equation can be expressed via elementary or special functions. So, in the case $\alpha_2=0$

$$\begin{aligned}
&\psi=Z_\nu(\sqrt{\beta^2+\alpha_1^2\gamma}\ \omega) && \text{if}\quad \beta^2+\alpha_1^2\gamma\neq 0,\quad \nu=\alpha_1\sqrt{-\gamma},\\
&\psi=C_1\omega^\beta+C_2\omega^{-\beta} && \text{if}\quad \beta^2+\alpha_1^2\gamma=0,\quad \beta\neq 0,\\
&\psi=C_1\ln\omega+C_2 && \text{if}\quad \beta=\gamma=0.
\end{aligned}$$

Here and below Z_ν is the general Bessel function of order ν, W is the Whittaker functions, C_0, C_1, C_2, and C_3 are arbitrary constants. In the case $\alpha_1=0$

$$\begin{aligned}
&\psi=Z_1(\sqrt{\beta^2+\alpha_2^2}\ \omega) && \text{if}\quad \gamma=0,\\
&\psi=\frac{1}{\omega}W\left(\frac{\beta^2+\alpha_2^2}{4\alpha_2\sqrt{-\gamma}};\frac{1}{2};\alpha_2\sqrt{-\gamma}\ \omega^2\right) && \text{if}\quad \gamma\neq 0.
\end{aligned}$$

2. $\langle\partial_t,\ D^x+\alpha D^t+\varkappa J_{12}+R(0,0,\mu t)+Z(\varepsilon_1),\ R(0,0,1)+Z(\varepsilon_2)\rangle$, where $\alpha\neq 0$, $\mu(\alpha-1)=0$, $\varepsilon_1(\alpha-1)=\varepsilon_2(2\alpha-1)=0$;

$$\begin{aligned}
&u^1=r^{-\alpha}(x_1\varphi^3-x_2(\varphi^1+\varkappa\varphi^3)),\quad u^1=r^{-\alpha}(x_2\varphi^3+x_1(\varphi^1+\varkappa\varphi^3)),\\
&u^3=r^{1-\alpha}\varphi^2+\mu\ln r,\quad p=r^{2-2\alpha}h+\varepsilon_1\ln r+\varepsilon_2x_3
\end{aligned}$$

where $r=(x_1^2+x_2^2)^{1/2}$, $\omega=\arctan x_2/x_1-\varkappa\ln r$, and new unknown functions $\varphi^a=\varphi^a(\omega)$ and $h=h(\omega)$ satisfy the reduced system

$$\begin{aligned}
&\varphi^1\varphi^1_\omega+(1-\alpha)\varphi^3\varphi^1+((1+\varkappa^2)\varphi^3+\varkappa\varphi^1)(\varphi^1+\varkappa\varphi^3)-2(1-\alpha)\varkappa h+(1+\varkappa^2)h_\omega=\varkappa\varepsilon_1,\\
&\varphi^1\varphi^2_\omega+(1-\alpha)\varphi^3\varphi^2+\mu\varphi^3+\varepsilon_2=0,\\
&\varphi^1\varphi^3_\omega+(1-\alpha)\varphi^3\varphi^3-(\varphi^1+\varkappa\varphi^3)^2+2(1-\alpha)h-\varkappa h_\omega+\varepsilon_1=0,\\
&\varphi^1_\omega+(2-\alpha)\varphi^3=0.
\end{aligned}\tag{4}$$

There exist three different cases of integration of system (4). If $\alpha=2$ then any solution of (4) belongs to a family from the following ones

$$\begin{aligned}
&\varphi^1=\varphi^2=0,\quad \varphi^3=C_1,\quad h=-\tfrac{1}{2}(1+\varkappa^2)C_1^2;\\
&\varphi^1=\varphi^3=h=0,\quad \varphi^2=\varphi^2(\omega);\\
&\varphi^1=C_1,\quad \varphi^2=C_2,\quad \varphi^3=0,\quad h=-\tfrac{1}{2}C_1^2;\\
&\varphi^1=C_1,\quad \varphi^2=C_2(\omega+C_3)^{-1},\quad \varphi^3=-C_1(\omega+C_3)^{-1},\quad h=(\varkappa(\omega+C_3)^{-1}-\tfrac{1}{2})C_1^2;\\
&\varphi^1=C_1,\quad \varphi^2=C_2\cos^{-1}(C_3\omega+C_4),\quad \varphi^3=C_1C_3\tan(C_3\omega+C_4),\\
&h=\tfrac{1}{2}C_1^2(C_3^2(1+\varkappa^2)-1)-\varkappa C_1\varphi^3;\\
&\varphi^1=C_1,\quad \varphi^2=\frac{C_2}{B_1e^{C_3\omega}+B_2e^{-C_3\omega}},\quad \varphi^3=-C_1C_3\frac{B_1e^{C_3\omega}-B_2e^{-C_3\omega}}{B_1e^{C_3\omega}+B_2e^{-C_3\omega}},\\
&h=-\tfrac{1}{2}C_1^2(C_3^2(1+\varkappa^2)+1)-\varkappa C_1\varphi^3.
\end{aligned}$$

In the case $\alpha=1$ we obtain that $\varepsilon_2=0$, $\varphi^2=\mu\ln\varphi^1+C_0$, $\varphi^3=-\varphi^1_\omega$, $h_\omega=\varkappa(\varphi^1\varphi^1_{\omega\omega}-(\varphi^1_\omega)^2)$, and $(1+\varkappa^2)\varphi^1_{\omega\omega}-2\varkappa\varphi^1_\omega+\varphi^1=\varepsilon_1(\varphi^1)^{-1}$. If additionally $\varepsilon_1=0$ then

$$\varphi^1=\left(C_1\cos\frac{\omega}{1+\varkappa^2}+C_2\sin\frac{\omega}{1+\varkappa^2}\right)\exp\frac{\varkappa\omega}{1+\varkappa^2},\quad h=-\frac{1}{2}\frac{C_1^2+C_2^2}{1+\varkappa^2}\exp\frac{2\varkappa\omega}{1+\varkappa^2}+C_3.$$

Let $\alpha \notin \{1;2\}$. Then $\varepsilon_1 = \mu = 0$, $\varepsilon_2(2\alpha-1) = 0$, $\varphi^3 = -(2-\alpha)^{-1}\varphi^1_\omega$,

$$h = -\frac{1}{2}\left(\frac{-2\varkappa}{2-\alpha}\varphi^1\varphi^1_\omega + \frac{1+\varkappa^2}{(2-\alpha)^2}(\varphi^1_\omega)^2 + (\varphi^1)^2 - \frac{C_0}{(1-\alpha)(2-\alpha)}(\varphi^1)^{2\frac{1-\alpha}{2-\alpha}}\right),$$

$$\varphi^2 = (\varphi^1)^{\frac{1-\alpha}{2-\alpha}}(C_3 + \varepsilon_2\textstyle\int(\varphi^1)^{-\frac{3-2\alpha}{2-\alpha}}d\omega),$$

$$(1+\varkappa^2)\varphi^1_{\omega\omega} - 2\varkappa(2-\alpha)\varphi^1_\omega + (2-\alpha)^2\varphi^1 = C_0(\varphi^1)^{-\frac{\alpha}{2-\alpha}}.$$

The last equation is easy solved if $C_0 = 0$.

3 $SO(3)$-partially invariant solutions of Euler equations

The concept of partially invariant solutions was introduced by Ovsiannikov [1] as a generalization of invariant solutions, which is possible for systems of partial differential equations (PDEs). The algorithm for finding partially invariant solutions is very difficult to apply. For this reason it is used more rarely than the classical Lie algorithm for constructing invariant solutions.

In this section we describe the process of constructing $SO(3)$-partially invariant solutions of the minimal defect which is equal to 1 for the given representation of $so(3)$ generated by the operators J_{ab} from $A(E)$ (2) (see [10] for detail).

A complete set of functionally independent invariants of the group $SO(3)$ in the space of the variables $(t,\vec{x},\vec{u},p)$ is exhausted by the functions t, $|\vec{x}|$, $\vec{x}\cdot\vec{u}$, $|\vec{u}|$, p, so any $SO(3)$-partially invariant solution of the minimal defect has the form

$$u^R = v(t,R),\ u^\theta = w(t,R)\sin\psi(t,R,\theta,\varphi),\ u^\varphi = w(t,R)\cos\psi(t,R,\theta,\varphi),\ p = p(t,R). \quad (5)$$

Hereafter for convenience the spherical coordinates are used. Substituting (5) into EEs (1), we obtain the system of PDEs for the functions v, w, ψ, p :

$$\begin{aligned}
&v_t + vv_R - R^{-1}w^2 + p_r = 0, \qquad w_t + vw_R + R^{-1}vw = 0,\\
&w(\psi_t + v\psi_R + R^{-1}w\psi_\theta\sin\psi + R^{-1}w\cos\psi(\sin\theta)^{-1}(\psi_\varphi - \cos\theta)) = 0,\\
&Rv_r + 2v + w\psi_\theta\cos\psi - (\sin\theta)^{-1}w\sin\psi(\psi_\varphi - \cos\theta) = 0.
\end{aligned} \quad (6)$$

It follows from (6) if $w=0$ that $v = \eta R^{-2}$, $p = \eta_t R^{-1} - \frac{1}{2}\eta^2R^{-4} + \chi$, where η and χ are arbitrary smooth functions of t. The corresponding solution of EEs

$$u^R = \frac{\eta}{R^2}, \qquad u^\theta = u^\varphi = 0, \qquad p = \frac{\eta_t}{R} - \frac{\eta^2}{2R^4} + \chi \quad (7)$$

is invariant with respect to $SO(3)$. Note that flow (7) is a solution of the Navier–Stokes equations too, and it is the unique $SO(3)$-partially invariant solutions of the minimal defect for them.

Below $w \neq 0$. Then two last equations of (6) form an overdetermined system in the function ψ. This system can be rewritten as follows

$$\begin{aligned}
&\psi_\theta + Rw^{-1}\sin\psi(\psi_t + v\psi_R) = -G\cos\psi,\\
&\psi_\varphi + Rw^{-1}\cos\psi(\psi_t + v\psi_R)\sin\theta = G\sin\psi\sin\theta + \cos\theta,
\end{aligned} \quad (8)$$

where $G = w^{-1}(Rv_R + 2v)$. The Frobenius theorem gives the compatibility condition of (8):

$$G_t + vG_R = R^{-1}w\left(1+G^2\right). \quad (9)$$

If condition (9) holds, system (8) is integrated implicitly and its general solution has the form

$$F(\Omega_1,\Omega_2,\Omega_3) = 0, \quad (10)$$

where F is an arbitrary function of Ω_1, Ω_2, and Ω_3,

$$\Omega_1 = \frac{\sin\theta\sin\psi - G\cos\theta}{\sqrt{1+G^2}}, \qquad \Omega_2 = \varphi + \arctan\frac{\cos\psi}{\cos\theta\sin\psi + G\sin\theta}, \qquad \Omega_3 = h(t,r),$$

$h = h(t,R)$ is a fixed solution of the equation $h_t + vh_R = 0$ such that $(h_t, h_R) \neq (0,0)$. Equation (10) can be solved with respect to ψ in a number of cases, for example, if either $F_{\Omega_1} = 0$ or $F_{\Omega_2} = 0$. Equation (9) and two first equation of (6) form the "reduced" system for the invariant functions v, w, and p. It can be represented as the union of the system

$$R^2 f_{tR} + f f_{RR} - (f_R)^2 = g, \quad R^2 g_t + f g_R = 0, \qquad f := R^2 v, \quad g := (Rw)^2, \tag{11}$$

for the functions v and w (this system can be also considered a system for the functions f and g) and the equation

$$p_R = -v_t - vv_R - R^{-1}w^2 \tag{12}$$

which is one for the function p if v and w are known. Therefore, to construct solutions for EEs, we are to carry out the following chain of actions: 1) to solve system (11); 2) to integrate (12) with respect to p; 3) to find the function ψ from (10).

Theorem 1. *The maximal Lie invariance algebra of (11) is the algebra*

$$\mathcal{A} = \langle\, \partial_t,\ D^R = R\partial_R + v\partial_v + w\partial_w,\ D^t = t\partial_t - v\partial_v - w\partial_w \,\rangle.$$

A complete set of $\mathcal{A}$-inequivalent one-dimensional subalgebras of $\mathcal{A}$ is exhausted by four algebras $\langle\, \partial_t \,\rangle$, $\langle\, D^R \,\rangle$, $\langle\, \partial_t + D^R \,\rangle$, $\langle\, D^t + \kappa D^R \,\rangle$. In [10] we constructed the corresponding ansatzes for the functions v and w as well as the reduced systems arising after substituting the ansatzes into (11). Two first reduced systems were integrated completely. We also found all the solutions of system (11), for which f and g are polynomial with respect to R.

4 Nonclassical symmetries of Euler equations

In this section we give results on Q-conditional symmetry [11, 12] of (1) with respect to single differential operator $Q = \xi^0(t,\vec{x},\vec{u},p)\partial_t + \xi^a(t,\vec{x},\vec{u},p)\partial_a + \eta^a(t,\vec{x},\vec{u},p)\partial_{u^a} + \eta^0(t,\vec{x},\vec{u},p)\partial_p$, which were firstly presented in [13].

Theorem 2. *Any operator Q of Q-conditional symmetry of the Euler equations (1) either is equivalent to a Lie symmetry operator of (1) or is equivalent* ($\mathrm{mod} A(E)$) *to the operator*

$$\widetilde{Q} = \partial_3 + \zeta(t,x_3,u^3)\partial_{u^3} + \chi(t)x_3\partial_p, \tag{13}$$

where $\zeta_{u^3} \neq 0$, $\zeta_3 + \zeta\zeta_{u^3} = 0$, $\zeta_t + (u^3\zeta + \chi x_3)\zeta_{u^3} + (\zeta)^2 + \chi = 0$.

It follows from Theorem 2 that there exist two classes of the possible reductions on one independent variable for EEs, namely, the Lie reductions and the reductions corresponding to operators of form (13). Lie reductions of EEs (1) to systems in three independent variables were investigated in [5]. An ansatz constructed with the operator $\widetilde{Q}$ has the following form:

$$u^1 = v^1, \quad u^2 = v^2, \quad u^3 = x_3 v^3 + \psi(t, v^3), \quad p = q + \tfrac{1}{2}\chi(t)x_3^2,$$

where $v^a = v^a(t,x_1,x_2)$, $q = q(t,x_1,x_2)$, the function $\psi = \psi(t,v^3)$ is a solution of the equation $\psi_t - ((v^3)^2 + \chi)\psi_{v^3} + v^3\psi = 0$. Substituting this ansatz into (1), we obtain the corresponding reduced system $(i,j = 1,2)$:

$$v^i_t + v^j v^i_j + q_i = 0, \quad v^3_t + v^j v^3_j + (v^3)^2 + \chi = 0, \quad v^j_j + v^3 = 0.$$

The analogous problem for the Navier–Stokes equations (NSEs)

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} + \nabla p - \nu \triangle \vec{u} = \vec{0}, \qquad \operatorname{div} \vec{u} = 0 \qquad (\nu \neq 0) \tag{14}$$

describing the motion of an incompressible viscous fluid was solved by Ludlow, Clarkson, and Bassom in [14]. Their result can be reformulated in the following manner: *any (real) operator Q of nonclassical symmetry of (14) is equivalent to a Lie symmetry operator of (14).* Therefore, all the possible reductions of NSEs on one independent variable are exhausted by the Lie reductions. The maximal Lie invariance algebra of NSEs (14) is similar to one of EEs (see [15, 16]):

$$A(\mathrm{NS}) = \langle \partial_t,\ J_{ab},\ D^t + \tfrac{1}{2} D^x,\ R(\vec{m}(t)),\ Z(\zeta(t)) \rangle.$$

The Lie reductions of NSEs were completely described in [17].

It should be noted that non-classical invariance of hydrodynamics equation (in particular, the Euler and Navier–Stokes equations) with respect to involutive families of two and three operators have not been investigated. The complete solving of this complicated problem would allow to describe all the possible reductions of the equations under considerations to systems of PDEs in two independent variables and to systems of ODEs.

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