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Boussineq-Type Equations and "Switching" Solitons

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It is well known that the Boussinesq equation is the *bidirectional* equivalent of the celebrated Korteweg-de Vries equation. Here we consider Boussinesq-type versions of two classical unidirectional integrable equations. A procedure is presented for deriving multisoliton solutions of one of these equations – a bidirectional Kaup–Kupershmidt equation. These solitons have the unusual property that they "switch" shape on switching their direction of propagation.

1 Introduction

In a recent article [1], we constructed a *bidirectional* version of the well-known Kaup–Kupershmidt (KK) equation [2, 3]

$$u_t + 45u^2u_x - \frac{75}{2}u_xu_{xx} - 15uu_{3x} + u_{5x} = 0, \tag{1}$$

which has the nonlocal form

$$5\,\partial_x^{-1}u_{tt} + 5u_{xxt} - 15uu_t - 15u\,\partial_x^{-1}u_t - 45u^2u_x + \frac{75}{2}u_xu_{xx} + 15uu_{3x} - u_{5x} = 0. \tag{2}$$

In Ref. [1], equation (2) was designated the bidirectional Kaup–Kupershmidt (bKK) equation. A second nonlinear evolution equation (NEE) that is also of interest here,

$$5\,\partial_x^{-1}u_{tt} + 5u_{xxt} - 15uu_t - 15u_x\partial_x^{-1}u_t - 45u^2u_x + 15u_xu_{xx} + 15uu_{3x} - u_{5x} = 0, \tag{3}$$

was formulated in [1] as a bidirectional counterpart of the classical Sawada–Kotera (SK) equation [4, 5]

$$u_t + 45u^2u_x - 15u_xu_{xx} - 15uu_{3x} + u_{5x} = 0. \tag{4}$$

The integrability of equations (2) and (3) was assured by finding their Lax pairs [1]. Indeed, by obtaining the bilinear form of equation (3), we were able to identify this bidirectional equation with the well-known Ramani equation [6] (see equation (7) below). The latter equation has been studied extensively – though only in its more familiar bilinear form (7) – and is now deemed to be completely integrable [6, 7, 8, 9]; we shall refer to equation (3) as the "bSK-Ramani" equation. On the other hand, the bKK equation (2) has received little attention of note in the literature (although the equation in its normal form (2) is listed in the Jimbo–Miwa classification of integrable systems [10]). In Ref. [1] we reported its Lax pair, along with an infinity of conservation laws. We also derived there the solitary-wave solution which has the remarkable property that it "switches" shape on switching its direction of propagation.

In this paper, a procedure is described for obtaining multisoliton solutions of the bKK equation (2). The preliminary results presented here build on the work of the prior study [1] where it was shown that the 'anomalous' character of these solitons arises quite naturally within Hirota's bilinear transform theory [11, 12]. Yet our approach also makes use of the strategy pursued by one of us (A.P.) to obtain the soliton solutions of its unidirectional cousin, the KK equation (1) [13]. However, the current problem is complicated by the need to take account of the bidirectional nature of the bKK solitons; like the solitary wave, they too are found to be *directionally dependent.*

2 Bilinear forms and solitary waves

Following Hirota [11], we make a change of dependent variable

$$u\,(x,t)=\alpha\,\partial_x^2 \ln f\,(x,t)\,,\qquad \alpha=\text{const.} \tag{5}$$

where ∂_x^n denotes the nth partial derivative with respect to x. Under this transformation, we find that the bSK-Ramani equation (3) has *two* bilinear forms [1]: when $\alpha=-1$ we get

$$\begin{aligned}&\left(80\mathrm{D}_t^2+20\mathrm{D}_x^3\mathrm{D}_t-\mathrm{D}_x^6\right)f\cdot f-\left(120\mathrm{D}_x\mathrm{D}_t-30\mathrm{D}_x^4\right)f\cdot g=0,\\ &\mathrm{D}_x^2 f\cdot f+2f\cdot g=0,\end{aligned} \tag{6}$$

where D_x, D_t are the usual Hirota derivatives [12]

$$\mathrm{D}_x\mathrm{D}_t\,a\,(x,t)\cdot b\,(x,t)=\left(\partial_x-\partial_{x'}\right)\left(\partial_t-\partial_{t'}\right)a\,(x,t)\,b\left(x',t'\right)\big|_{x'=x,\ t'=t}$$

and $g\,(x,t)$ is an auxiliary function. The second bilinear form has $\alpha=-2$ and is given by

$$\left(5\mathrm{D}_t^2+5\mathrm{D}_x^3\mathrm{D}_t-\mathrm{D}_x^6\right)f\cdot f=0. \tag{7}$$

The single bilinear equation (7) identifies the bSK-Ramani equation (3) with Ramani's equation [6], whereas the less well-known coupled system (6) appeared somewhat later [10].

Similarly, under the transformation (5), the bKK equation (2) admits two bilinear representations [1]: $\alpha=-1$

$$\begin{aligned}&\left(80\mathrm{D}_t^2+20\mathrm{D}_x^3\mathrm{D}_t-\mathrm{D}_x^6\right)f\cdot f-120\mathrm{D}_x\mathrm{D}_t f\cdot g+30\mathrm{D}_x^2 f\cdot h=0,\\ &\mathrm{D}_x^2 f\cdot f+2f\cdot g=0,\\ &\mathrm{D}_x^4 f\cdot f+2f\cdot h=0;\end{aligned} \tag{8}$$

$\alpha=-2$:

$$\begin{aligned}&16\left(5\mathrm{D}_t^2+5\mathrm{D}_x^3\mathrm{D}_t-\mathrm{D}_x^6\right)f\cdot f-30\mathrm{D}_x^4 f\cdot g+30\mathrm{D}_x^2 f\cdot h=0,\\ &\mathrm{D}_x^2 f\cdot f+f\cdot g=0,\\ &\mathrm{D}_x^4 f\cdot f+f\cdot h=0.\end{aligned} \tag{9}$$

Equations (8) and (9), in which g and h are auxiliary functions, are derived in Ref. [1].

Finding the multisoliton solutions of the bSK-Ramani equation (3) is straightforward since we may solve the *single* bilinear form (7) rather than the coupled system (6). Thus, the N-soliton solution of equation (7) is given by Hirota's ansatz [11]

$$f\,(x,t)=\sum_{\mu=0,1}\exp\left[\sum_{i=1}^{N}\mu_i\theta_i+\sum_{1\le i<j\le N}\mu_i\mu_j\ln A_{ij}\right], \tag{10}$$

where $\theta_i=p_ix+\omega_it+\eta_i$ $(i=1,\ldots,N)$ and p_i, ω_i, η_i are constant parameters. Following Ref. [13], we will call the generic solution (10) the *regular* N-soliton: observe that it is described by a single interaction coefficient A_{ij}. The solitary wave is given by setting $N=1$ in equation (10) and yields the familiar sech^2 pulse [1]

$$u\,(x,t)=-\frac{1}{2}\,p^2\mathrm{sech}^2\frac{1}{2}\left(px+\omega t+\eta\right), \tag{11}$$

where $\omega(p)$ satisfies the quadratic dispersion relation

$$5\omega^2+5\omega p^3-p^6=0. \tag{12}$$

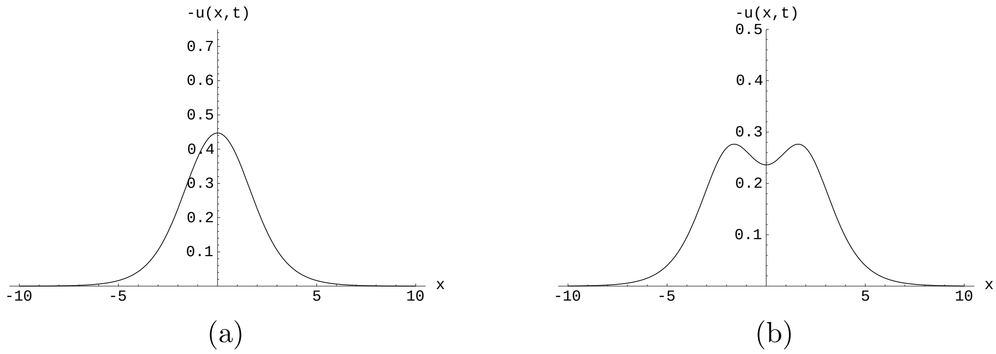


Figure 1. Solitary-wave solutions of the bKK equation: (a) a right-travelling single-humped wave, (b) a left-travelling double-humped wave.

For the bKK equation, no reduction of the bilinear forms (8) and (9) to a single bilinear equation, akin to the Ramani equation (7), is possible. We must therefore solve one or other of the coupled systems (8) or (9) for which no prescribed ansatz, comparable to the regular N-soliton (10), is available. However, we may exploit the close connection between the bKK and bSK-Ramani equations – that is evident from equations (2) and (3) – to argue as follows. Since the bilinear forms (6) and (7) of the bSK-Ramani equation are equivalent under $f^2 \leftrightarrow f$ [1], the N-soliton solution of the coupled bilinear form (6) is the *square* of the regular N-soliton (10). But then the duality of the bKK and bSK-Ramani equations suggests the following hypothesis: the N-soliton solution of the bKK bilinear form (8) will *mimic* its counterpart for the corresponding bSK-Ramani system (6). For example, if we apply this reasoning to the regular solitary wave (set $N = 1$ in (10)), we obtain the solution of equation (8) [1],

$$f(x,t) = 1 + e^{\theta} + \frac{1}{16}a^2 e^{2\theta}, \qquad \theta = px + \omega t + \eta, \tag{13}$$

where

$$a^2 = \frac{4\omega - p^3}{\omega - p^3} \qquad (a > 0) \tag{14}$$

and $\omega(p)$ satisfies the (bSK-Ramani) dispersion relation (12). Then, using $u = -\partial_x^2 \ln f$ (equation (5) with $\alpha = -1$), we obtain the 'anomalous' solitary wave of the bKK equation (2)

$$u(x,t) = -ap^2 \frac{a + 2\cosh\theta}{(a\cosh\theta + 2)^2}, \tag{15}$$

which was first reported in Ref. [1]. The most significant feature of this solitary wave is that *it "switches" its shape on switching direction* (cf. the bSK-Ramani solitary wave (11) that propagates to the left or right with the *same* bell-shaped profile). The right-travelling single-humped solitary wave is shown in Fig. 1(a), whilst the left-running wave has the double-humped profile pictured in Fig. 1(b) (where here, and in subsequent figures, we plot the physical wave $-u(x,t)$). Extending the argument, we conjecture that the N-soliton of the bKK equation (8) has the structure – though not the precise analytical form – of the *squared* regular N-soliton (10). We shall use this duality hypothesis – which was formulated in Ref. [1] – to obtain higher-order soliton solutions of the bKK equation; in effect, we choose to solve the coupled bilinear form (8) rather than the alternate system (9).

3 Two-soliton solution of the bKK equation

Before proceeding, it will be helpful to introduce the following notation: if $F(\mathrm{D}_x, \mathrm{D}_t)$ is any bilinear operator, then we define $F(\boldsymbol{p}) = F(p, \omega)$. Now, the regular two-soliton solution of the bSK-Ramani equation is given by (set $N = 2$ in equation (10))

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + A_{12}e^{\theta_1+\theta_2}, \qquad \theta_i = p_i x + \omega_i t + \eta_i, \qquad i = 1, 2, \tag{16}$$

which solves equation (7) if

$$A_{12} = -\frac{F_{\mathrm{R}}(\boldsymbol{p}_1 - \boldsymbol{p}_2)}{F_{\mathrm{R}}(\boldsymbol{p}_1 + \boldsymbol{p}_2)} = -\frac{5(\omega_1-\omega_2)^2 + 5(\omega_1-\omega_2)(p_1-p_2)^3 - (p_1-p_2)^6}{5(\omega_1+\omega_2)^2 + 5(\omega_1+\omega_2)(p_1+p_2)^3 - (p_1+p_2)^6} \tag{17}$$

and $\omega_i(p_i)$ satisfies the dispersion relation (cf. equation (12))

$$F_{\mathrm{R}}(\boldsymbol{p}_i) = 5\omega_i^2 + 5\omega_i p_i^3 - p_i^6 = 0, \qquad i = 1, 2, \tag{18}$$

where $F_{\mathrm{R}}(\mathrm{D}_x, \mathrm{D}_t) = 5\mathrm{D}_t^2 + 5\mathrm{D}_x^3\mathrm{D}_t - \mathrm{D}_x^6$ is the Ramani bilinear operator. According to our duality hypothesis, the two-soliton solution of the bKK equation will mimic f^2 (a solution of the bSK-Ramani bilinear form (6)). We therefore seek a solution of the bilinear form (8) with

$$\begin{aligned} f(x,t) = {} & 1 + e^{\theta_1} + e^{\theta_2} + \frac{1}{16}a_1^2\, e^{2\theta_1} + \frac{1}{16}a_2^2\, e^{2\theta_2} + b_{12}\, e^{\theta_1+\theta_2} \\ & + \frac{A}{16}\left(a_1^2\, e^{2\theta_1+\theta_2} + a_2^2\, e^{\theta_1+2\theta_2}\right) + \left(\frac{A}{16}\right)^2 a_1^2\, a_2^2\, e^{2(\theta_1+\theta_2)}, \end{aligned} \tag{19}$$

where (cf. equation (14))

$$a_i^2 = \frac{4\omega_i - p_i^3}{\omega_i - p_i^3}, \qquad i = 1, 2, \tag{20}$$

and $\omega_i(p_i)$ satisfies the (bSK-Ramani) dispersion law (18). The expression (19) merits further comment: firstly, it has been *normalised* by setting the coefficients of the terms e^{θ_i} to unity (η_i are arbitrary). Further, f is symmetrical under the exchange $\theta_1 \leftrightarrow \theta_2$. Finally, by applying the "elastic" interaction property of colliding solitons [14, 15] – whereby (19) separates asymptotically into two distinct 'solitary' waves of the form (13)–(14) – we are left with just the two unknown constants b_{12} and A. The parameter A arises quite naturally as a measure of the post-interaction phase shifts of the constituent solitary waves, and so plays the same rôle as A_{12} in the bSK-Ramani two-soliton (16).

We now substitute the putative bKK two-soliton (19) into the bilinear form (8) and make use of the standard result [12]

$$F(\mathrm{D}_x, \mathrm{D}_t)\, e^{\theta_1} \cdot e^{\theta_2} = F(\boldsymbol{p}_1 - \boldsymbol{p}_2)\, e^{\theta_1+\theta_2}, \qquad \theta_i = p_i x + \omega_i t + \eta_i, \quad i = 1, 2.$$

Following some routine but lengthy algebra (that is best carried out using a symbolic manipulation programme such as Mathematica), we find that $A = A_{12}$, equation (17), and

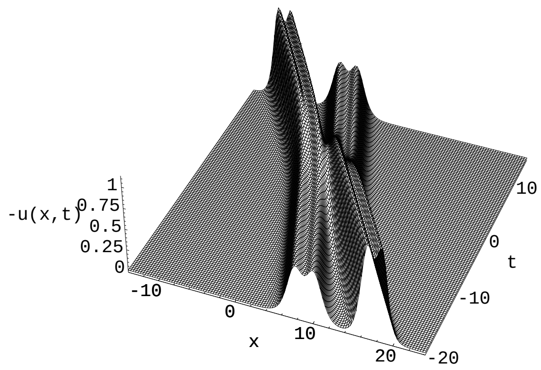
$$b_{12} = \frac{\Delta_{12}}{2F_{\mathrm{R}}(\boldsymbol{p}_i + \boldsymbol{p}_j)} = \frac{\Delta_{12}}{2D_{12}}, \tag{21}$$

where

$$\Delta_{12} = 20\omega_1\omega_2 + 10\omega_1 p_2\left(3p_1^2 + p_2^2\right) + 10\omega_2 p_1\left(p_1^2 + 3p_2^2\right) - p_1 p_2\left(12p_1^4 - 5p_1^2p_2^2 + 12p_2^4\right) \tag{22}$$

and

$$\begin{aligned} D_{12} = {} & 10\omega_1\omega_2 + 5\omega_1 p_2\left(3p_1^2 + 3p_1p_2 + p_2^2\right) + 5\omega_2 p_1\left(p_1^2 + 3p_1p_2 + 3p_2^2\right) \\ & - p_1p_2\left(6p_1^4 + 15p_1^3p_2 + 20p_1^2p_2^2 + 15p_1p_2^3 + 6p_2^4\right). \end{aligned} \tag{23}$$



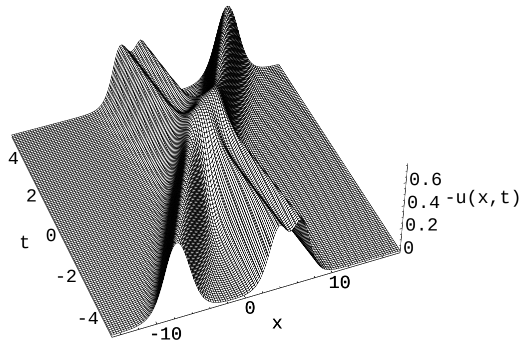


Figure 2. A perspective view of the bKK two-soliton: (a) the interaction of two left-travelling double-humped solitary waves, (b) the head-on collision of a single- and a double-humped pulse.

This completes the derivation of the two-soliton solution $u(x,t)$ of the bKK equation (2) which is obtained explicitly from (19) (with $A \to A_{12}$) and the relation $u = -\partial_x^2 \ln f$. Fig. 2(a) shows a two-soliton comprised of a pair of double-humped 'solitary' waves propagating to the left, whilst Fig. 2(b) pictures the head-on collision between a single-peaked and a double-peaked 'solitary' wave. Typically, the soliton pulses emerge from the interactions intact, except for the clearly visible phase shifts. The bKK two-soliton (19) bears further comment. It shares the same wave dynamics as the bSK-Ramani two-soliton, equation (16): their colliding solitary waves undergo identical phase shifts that are determined by the common interaction coefficient A_{12}, equation (17). This bears out the intimate connection between the bKK and bSK-Ramani equations that is already apparent through the shared dispersion relations (12) and (18), and justifies the duality hypothesis on which our solution procedure is based. Another important feature of (19) is the 'new' parameter b_{12}, equation (21), which cannot be expressed in terms of A_{12} alone (cf. the bSK-Ramani two-soliton (16)). It is instructive to compare this key parameter with its counterpart for the bSK-Ramani equation. Squaring (and normalising) the regular two-soliton (16), and extracting the coefficient of $\mathrm{e}^{\theta_1+\theta_2}$, we find

$$b_{12}^{\mathrm{R}} = \frac{1}{2}(A_{12}+1) = \frac{\Delta_{12}^{\mathrm{R}}}{2D_{12}}$$

with

$$\Delta_{12}^{\mathrm{R}} = 20\omega_1\omega_2 + 10\omega_1 p_2\left(3p_1^2+p_2^2\right) + 10\omega_2 p_1\left(p_1^2+3p_2^2\right) - p_1p_2\left(12p_1^4+40p_1^2p_2^2+12p_2^4\right).$$

Thus, b_{12} mimics b_{12}^{R} (they differ only in the $p_1^3p_2^3$ term in their numerators) and suggests that our duality hypothesis can be extended to include this crucial parameter. This further conjecture will help us when we seek solitons of higher order.

4 Further soliton solutions of the bKK equation

For the sake of brevity, we must content ourselves with describing the main results. We will leave a more complete presentation of these preliminary results – giving a full account of the technical details – to a future work.

According to our duality hypothesis, to obtain the bKK three-soliton solution we start with the regular three-soliton (put $N = 3$ in equation (10)). We then square (and normalise) this expression, introducing a minimal number of undetermined coefficients to give the form of the ansatz f (consistent with the symmetry in θ_i, $i = 1, 2, 3$). Rather than solve the coupled bilinear form (8) directly, we proceed by iteration on the solitons of lower order. (This soliton reduction procedure was first developed in Ref. [13] to solve the related KK equation (1)). Once f has been reduced to a solitary wave, equation (13), and then a two-soliton, equation (19), we arrive at the three-soliton

$$\begin{aligned} f = 1 &+ \sum_{i=1}^{3} \mathrm{e}^{\theta_i} + \frac{1}{16}\sum_{i=1}^{3} a_i^2 \mathrm{e}^{2\theta_i} + \sum_{1\le i<j\le 3} b_{ij}\,\mathrm{e}^{\theta_i+\theta_j} + \frac{1}{16}\sum_{1\le i<j\le 3} A_{ij}\left(a_i^2\mathrm{e}^{2\theta_i+\theta_j} + a_j^2\mathrm{e}^{\theta_i+2\theta_j}\right) \\ &+ b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3} + \frac{1}{16^2}\sum_{1\le i<j\le 3} A_{ij}^2 a_i^2 a_j^2\,\mathrm{e}^{2(\theta_i+\theta_j)} \\ &+ \frac{1}{16}\left[a_1^2 b_{23}A_{12}A_{13}\mathrm{e}^{2\theta_1+\theta_2+\theta_3} + a_2^2 b_{13}A_{12}A_{23}\mathrm{e}^{\theta_1+2\theta_2+\theta_3} + a_3^2 b_{12}A_{13}A_{23}\mathrm{e}^{\theta_1+\theta_2+2\theta_3}\right] \\ &+ \frac{1}{16^2}A_{12}A_{13}A_{23}\left[a_1^2a_2^2A_{12}\mathrm{e}^{2(\theta_1+\theta_2)+\theta_3} + a_1^2a_3^2A_{13}\mathrm{e}^{2\theta_1+\theta_2+2\theta_3} + a_2^2a_3^2A_{23}\mathrm{e}^{\theta_1+2(\theta_2+\theta_3)}\right] \\ &+ \frac{1}{16^3}a_1^2a_2^2a_3^2A_{12}^2A_{13}^2A_{23}^2\,\mathrm{e}^{2(\theta_1+\theta_2+\theta_3)} \end{aligned} \tag{24}$$

in which all but one of the coefficients have been fixed. The parameters A_{ij} and b_{ij} in (24) generalise equations (17) and (21), respectively, in the obvious way. The only unknown is the ‘new’ parameter b_{123} which cannot be found by reducing f to a soliton of lower order. However, we can deduce the following useful reductions in this way: with $\boldsymbol{p}_i = (p_i, \omega_i)$, we have

$$b_{123}(\boldsymbol{p}_1,\boldsymbol{p}_2,\mathbf{0}) = b_{12}(\boldsymbol{p}_1,\boldsymbol{p}_2), \quad b_{123}(\boldsymbol{p}_1,\mathbf{0},\mathbf{0}) = b_{23}(\mathbf{0},\mathbf{0}), \quad b_{123}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_2) = \frac{1}{8}a_2^2A_{12}. \tag{25}$$

We now invoke our further conjecture that b_{123} will mimic its counterpart b_{123}^{R}: this yields

$$b_{123} = \frac{\Delta_{123}}{4D_{123}}, \qquad D_{123} = D_{12}D_{13}D_{23}, \tag{26}$$

where D_{ij} generalises (23) and

$$\begin{aligned} \Delta_{123} = 18 &\ll p_i^2p_j^2p_k^2\left(5\omega_i - 2p_i^3\right)\left(5\omega_j - 2p_j^3\right)\Delta_{ij}\gg +810 \ll p_i^{10}p_j^4p_k^4\gg +324p_1^6p_2^6p_3^6 \\ &- 4050 \ll \omega_i p_i^7p_j^4p_k^4\gg +405 \ll \omega_i p_i^5p_j^6p_k^4\gg -2430 \ll \omega_i p_i^3p_j^6p_k^6\gg \\ &+ 1620 \ll \omega_i p_i p_j^{10}p_k^4\gg -8100 \ll \omega_i\omega_j p_i^7p_jp_k^4\gg -4050 \ll \omega_i\omega_j p_i^5p_j^3p_k^4\gg \\ &+ 810 \ll \omega_i\omega_j p_i^5p_jp_k^6\gg +16200 \ll \omega_i\omega_j p_i^3p_j^3p_k^6\gg +3240 \ll \omega_i\omega_j p_ip_jp_k^{10}\gg \\ &- 16200 \ll \omega_i\omega_j\omega_k p_i^7p_jp_k\gg -8100 \ll \omega_i\omega_j\omega_k p_i^5p_j^3p_k\gg -81000\omega_1\omega_2\omega_3p_1^3p_2^3p_3^3. \end{aligned} \tag{27}$$

The symbol $\ll\gg$ denotes the sum over all distinct permutations of $(1,2,3)$ assigned to the subscripts (i,j,k) of the enclosed product, and Δ_{ij} generalises (22). All but two of the coefficients in (27) are fixed by the reductions (25); the remaining two coefficients are obtained by using the bilinear equation in (8) once more (though with a much simplified ansatz in place of (24)). The explicit three-soliton solution $u(x,t)$ of the bKK equation (2) follows from (24) and $u = -\partial_x^2 \ln f$. Fig. 3 shows a three-soliton solution in which two left-running double-humped ‘solitary’ waves collide head-on with a single-peaked pulse propagating to the right. Though we shall not do so here, we could continue in the same way to derive the four-soliton solution by iterating on the first three known solitons. In principle, we are now able to generate the N-soliton solution of the bKK equation (2) by iteration on the solitons of lower order; however, the practical difficulties should not be underestimated. The sheer complexity of the algebraic expressions involved will present severe difficulties beyond the first few multisolitons (even with the aid of symbolic software such as Mathematica).

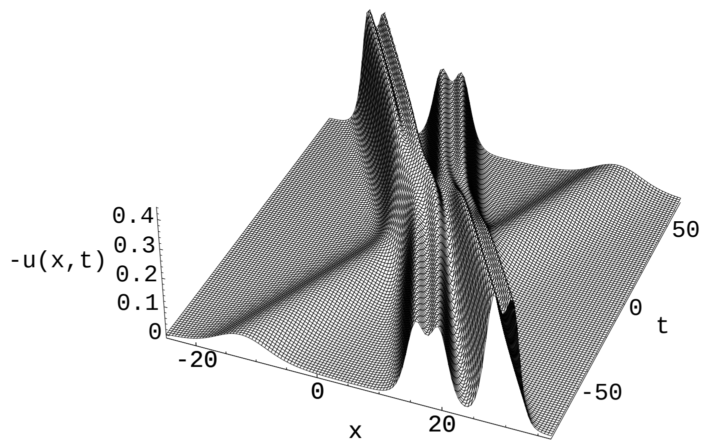


Figure 3. A three-soliton solution of the bKK equation showing the head-on interaction of a right-running single-peaked pulse with two left-running double-humped solitary waves.

5 Concluding remarks

A direct method has been presented for obtaining explicit multisoliton solutions of the bidirectional Kaup-Kupershmidt equation (2). Not surprisingly, these solitons possess the same remarkable property as the 'anomalous' solitary wave found in Ref. [1]; namely, their wave profiles are *directionally dependent*. As far as we know, this type of soliton behaviour has not been observed before now and these "switching" solitons are reported here for the first time. The 'anomalous' character of the bKK solitons – whose description requires the introduction of a new parameter at each order – arises quite naturally within the bilinear formalism as a *squared* regular N-soliton. This canonical form, in conjunction with the duality of the bKK and bSK-Ramani equations, provides the basis for the iterative procedure that is used to obtain the solitons of higher order. From a wave perspective, this formulation – couched in terms of the common interaction parameters A_{ij} and shared dispersion laws (18) – would seem to be the natural one. For not only does it make explicit the dynamical duality of the soliton solutions of the bKK and bSK-Ramani equations, but it also underlines the intimacy between these fundamentally different integrable bidirectional equations. This mirrors the deep connection between their better known undirectional cousins the KK equation (1) and the SK equation (4), respectively [2, 13, 16, 17]. We intend to report a more comprehensive discussion of these preliminary findings, together with the derivation of further multisolitons, in the near future.

[1] Dye J.M. and Parker A., On bidirectional fifth-order nonlinear evolution equations, Lax pairs, and directionally dependent solitary waves, *J. Math. Phys.*, 2001, V.42, 2567–2589.

[2] Kaup D.J., On the inverse scattering problem for the cubic eigenvalue problem of the class $\phi_{xxx} + 6Q\phi_x + 6R\phi = \lambda\phi$, *Stud. Appl. Math.*, 1980, V.62, 189–216.

[3] Kupershmidt B.A., A super KdV equation: an integrable system, *Phys. Lett. A.*, 1984, V.102, 213–215.

[4] Sawada K. and Kotera T., A method for finding N-soliton solutions of the KdV equation and KdV-like equation, *Prog. Theort. Phys.*, 1974, V.51, 1355–1367.

[5] Caudrey P.J., Dodd R.K. and Gibbon J.D., A new hierarchy of Korteweg-de Vries equations, *Proc. R. Soc. Lond. A.*, 1976, V.351, 407–422.

[6] Ramani A., Inverse scattering. Ordinary differential equations of Painlevé-type, and Hirota's bilinear formalism, *Ann. N.Y. Acad. Sci.*, 1981, V.373, 54–67.

[7] Hu X-B., Hirota-type equations, soliton solutions, Bäcklund transformations and conservation laws, *J. Partial Diff. Equs.*, 1990, V.3, 87–95.

[8] Hirota R., Soliton solutions of the BKP equations. II. The integral equation, *J. Phys. Soc. Jpn.*, 1989, V.58, 2705–2712.

[9] Hietarinta J., A search for bilinear equations passing Hirota’s three-soliton condition. I. KdV-type bilinear operators, *J. Math. Phys.*, 1987, V.28, 1732–1742.

[10] Jimbo M. and Miwa T., Solitons and infinite dimensional Lie algebras, *Publ. RIMS, Kyoto Univ.*, 1983, V.19, 943–1001.

[11] Hirota R., Direct methods in soliton theory, in Solitons, Editors R.K. Bullough and P.J. Caudrey, Topics in Current Physics, Berlin, Springer-Verlag, 1980, V.17, 157–176.

[12] Matsuno Y., Bilinear transformation method, Orlando, Academic Press, 1984.

[13] Parker A., On soliton solutions of the Kaup–Kupershmidt equation. II. ‘Anomalous’ N-soliton solutions, *Physica D*, 2000, V.137, 34–48.

[14] Ablowitz M.J. and Segur H., Solitons and the inverse scattering transform, Philadelphia, SIAM, 1981.

[15] Drazin P.G. and Johnson R.S., Solitons: an introduction, Cambridge, Cambridge University Press, 1990.

[16] Parker A., On soliton solutions of the Kaup–Kupershmidt equation. I. Direct bilinearisation and solitary wave, *Physica D*, 2000, V.137, 25–33.

[17] Fordy A.P. and Gibbons J., Some remarkable nonlinear transformations, *Phys. Lett. A*, 1980, V.75, 325.