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Extended SUSY with Central Charges in Quantum Mechanics

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We present a new form of supersymmetric quantum mechanics which is characterized by presence of non-trivial central charges. We show that the corresponding extended SUSY appears in a number of popular quantum mechanical models.

1 Introduction

Supersymmetric quantum mechanics (SUSY QM) [1] appeared first as a toy model for better understanding of SUSY itself. However, it turns out that SUSY QM systems themselves are so rich in structure and deal with new properties which became recently subjects per se studied by many mathematicians and theoretical physicists. The basic problems and various applications of SUSY quantum mechanics are discussed in a number of papers, refer, e.g., to surway [2].

Soon it appears that SUSY QM systems can admit more than two supercharges and so to have richer symmetry called extended SUSY. Such extended SUSY has good physical grounds, since there exists a number of realistic physical systems which admit more than two supercharges, see e.g., [3, 4, 6, 7]. Moreover, it was demonstrated in [3, 6, 7] that the Schrödinger–Pauli and the Dirac equations admit not only extended SUSY but also rather large algebras of discrete involutive symmetries isomorphic to $gl(4,\mathbb{C})$ and $gl(8,\mathbb{R})$ respectively. Thus it seams that extended SUSY is closely connected to discrete symmetries.

In the present paper we continue in our investigations [3, 6, 7] and study QM systems which admit extended SUSY. Moreover, we consider generalized extensions of symmetry superalgebras generated by additional supercharges and even operators as well. We prove that the Coulomb, Aharonov–Bohm–Colomb (ABC) and Aharonov–Casher systems admit extended SUSY with six supercharges and central charge and, besides, they admit extended algebras of discrete symmetries isomorphic to $gl(8,\mathbb{R})$. All mentioned symmetries are responsible for degeneracy of the corresponding energy spectra.

We introduce the concept of general SUSY QM systems with central charges, and prove that many popular quantum mechanical models are perfect examples of them.

2 Quantum mechanics with extended SUSY

We say that the Schrödinger type equation

$$H\psi = E\psi \tag{1}$$

is supersymmetric and has $N = 2n$ SUSY, if it admits a set of integrals of motion $Q_1, Q_2, \ldots, Q_n$ which commute with Hamiltonian H and satisfy the following relations

$$\{Q_a, \bar{Q}_b\} = Q_a\bar{Q}_b + \bar{Q}_bQ_a = 2\delta_{ab}H, \qquad a, b = 1, 2, \ldots, n,$$
$$\{Q_a, Q_b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0 \tag{2}$$

with δ_{ab} being the Kronecker symbol and $\bar{Q} = Q^\dagger$.

For $n = 1$ we recognize in (2) the Witten superalgebra which is characteristic algebra appearing in SUSY QM models. This algebra contains two odd elements (supercharges) Q_1 and $\bar{Q}_1$ and the only even element H, thus in this case we have $N = 2$ SUSY. For $n > 1$ one has a QM model with the so-called extended SUSY. Realistic QM models admitting extended SUSY are discussed in [3, 4, 6, 7].

Of course, relations (2) admit a formal generalization to the case when the number of even elements is larger than 1. Then the corresponding defining relations can be transformed to the following ones:

$$\{Q_a, \bar{Q}_b,\} = 2\delta_{ab}H + Z_{ab}, \qquad a, b = 1, 2, \ldots, n,$$
$$\{Q_a, Q_b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0, \tag{3}$$

where Z_{ab} are the so called *central charges* which commute with all elements Q_a, $\bar{Q}_a$, H of the superalgebra.

We shall show in Sections 3, 4 that such a generalization appears naturally for some popular QM problems.

3 Extended SUSY for the Coulomb problem

First we shall consider the free Dirac equation

$$(\gamma_\mu p^\mu - m)\psi(x) = 0, \tag{4}$$

where $p_\mu = i\frac{\partial}{\partial x^\mu}$, $\mu = 0, 1, 2, 3$, $x = (x_0, x_1, x_2, x_3)$, γ_μ are the Dirac matrices.

It was shown in [3, 6] that equation (4) admits a 64-dimensional algebra of involutive discrete symmetries. Basis elements of this algebra can be chosen in the following form

$$\Gamma_m, \quad \Gamma_m\Gamma_n, \quad \Gamma_m\Gamma_n\Gamma_p, \quad I_4, \tag{5}$$

were $m, n, p = 0, 1, \ldots, 6$, I_4 is the 4×4 unit matrix,

$$\Gamma_\mu = i\gamma_4\gamma_\mu\hat{\theta}_\mu \quad \text{(no sum over } \mu\text{)}, \qquad \Gamma_4 = i\gamma_4\hat{\theta}, \qquad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3,$$
$$\Gamma_5 = \gamma_4\gamma_2 c\theta, \qquad \Gamma_6 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5, \tag{6}$$

$\hat{\theta}$ and c are reflection and complex conjugation operators defined by the following relations:

$$\hat{\theta}_\mu\psi(x) = \psi(\theta_\mu x), \qquad \hat{\theta}\psi(x) = \psi(-x), \qquad c\psi(x) = \psi^*(x),$$
$$\theta_0 x = (-x_0, x_1, x_2, x_3), \qquad \theta_1 x = (x_0, -x_1, x_2, x_3), \qquad \theta_2 x = (x_0, x_1, -x_2, x_3),$$
$$\theta_3 x = (x_0, x_1, x_2, -x_3).$$

Operators (5) transform solutions of the Dirac equation into solutions and form a Lie algebra isomorphic to $gl(8, \mathbb{R})$. We notice that the set of operators (5) include reflections Γ_μ, Γ_4 and pure rotations $\Gamma_\mu\Gamma_\nu$ as well.

The Dirac equation with non-trivial potentials

$$L\psi \equiv (\gamma^\mu \pi_\mu - m)\,\psi = 0, \qquad \pi_\mu = p_\mu - eA_\mu \tag{7}$$

does not admit all symmetry operators (5) but only a part of them instead. Nevertheless, we shall show that for some vector-potentials A_μ equation (7) admits extra symmetries which form basses of extended algebras isomorphic to (5).

As an example consider the relativistic Coulomb system described by the Dirac equation (7) with

$$A_1 = A_2 = A_3 = 0, \qquad eA_0 = \frac{\alpha}{|x|} \tag{8}$$

and $|x| = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$.

Equation (7), (8) admits a specific integral of motion discovered by Johnson and Lippman [8]. We present this constant of motion in the following form

$$\hat{Q} = m\alpha \frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{|x|} + iD\left(\boldsymbol{\sigma}\cdot\boldsymbol{p} + i\gamma_4 \frac{\alpha}{|x|}\right). \tag{9}$$

Here $D = \gamma_0\left(\boldsymbol{\sigma}\cdot\boldsymbol{J} - \frac{1}{2}\right)$ with $\boldsymbol{J} = \boldsymbol{x}\times\boldsymbol{p} + \boldsymbol{\sigma}/2$ is the Dirac constant of motion, $\boldsymbol{\sigma} = i\boldsymbol{\gamma}\times\boldsymbol{\gamma}/2$.

Operators $\hat{Q}$ and D commute with the Dirac Hamiltonian $H = \gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p} + \gamma_0 m + \frac{\alpha}{|x|}$ and anticommute among themselves. They are odd elements of the five dimensional superalgebra which contains also three even elements, namely H, $\hat{Q}^2$ and D^2. The commutation relations for odd-even and even-even elements have the form $[A,B] = 0$.

We notice that eigenvalues of Hamiltonian H can be expressed via eigenvalues of D and $\hat{Q}$. Indeed, using the relations

$$D^2 = J^2 + \frac{1}{4}, \qquad Q^2 = D^2\left(H^2 + m^2\right) - \alpha^2 m^2$$

and denoting eigenvalues of mutually commuting operators D^2, Q^2 and H by κ^2, q^2 and E respectively, we obtain the following relation

$$E^2 = \frac{q^2}{\kappa^2} + m^2\left(1 - \frac{\alpha^2}{\kappa^2}\right), \qquad \kappa = 0, 1, 2, \ldots.$$

Using this expression we shall demonstrate that the Coulomb system defined in (7)and (8) admits extended superalgebra which include six supercharges Q_a, $\bar{Q}_a$, $a = 1, 2, 3$ and one central charge $Z_{ab} = \delta_{ab} Z$ where

$$\begin{aligned}
&Q_1 = (1 + i\Gamma_5\Gamma_1\Gamma_2)\hat{Q}, && \bar{Q}_1 = (1 - i\Gamma_5\Gamma_1\Gamma_2)\hat{Q},\\
&Q_2 = i(\Gamma_1 + \Gamma_5)\Gamma_2\Gamma_3)\hat{Q}, && \bar{Q}_2 = i(\Gamma_1 - G_5)\Gamma_2\Gamma_3)\hat{Q},\\
&Q_3 = \Gamma_5(1 + i\Gamma_1\Gamma_3)\hat{Q}, && \bar{Q}_3 = \Gamma_5(1 - i\Gamma_1\Gamma_3)\hat{Q},\\
&Z = \left(\alpha^2 - D^2\right) m^2/\kappa^2, && \hat{H} = H^2.
\end{aligned} \tag{10}$$

Using the relations

$$[\Gamma_k, H] = [\Gamma_k, D] = 0, \qquad \{\Gamma_k, Q\} = \{\Gamma_5, \Gamma_a\} = \{\Gamma_5, i\} = 0, \qquad \{\Gamma_a, \Gamma_b\} = 2\delta_{ab},$$

where $k = 1, 2, 3, 5$, $a, b = 1, 2, 3$, we find that operators (10) commute with H and satisfy superalgebra (3). Thus, *the Coulomb system admits $N = 6$ extended SUSY with non-trivial central charge.* This symmetry algebra is closely related to the 64-dimensional algebra of involutive symmetries described in [3, 6]. Indeed, for any $q \neq 0$ we can define the following symmetry operators of the stationary Dirac equation

$$\hat{\Gamma}_0 = i\Gamma_1\Gamma_2\Gamma_3, \qquad \hat{\Gamma}_a = (Q_a + \bar{Q}_a)/2q, \qquad \hat{\Gamma}_{3+a} = (Q_a - \bar{Q}_a)/2iq, \qquad a = 1, 2, 3 \tag{11}$$

which satisfy

$$\{\hat{\Gamma}_K, \hat{\Gamma}_N\} = 2g_{KN}, \qquad K, N = 0, 1, \ldots, 6.$$

The only nonzero elements of tensor g_{KN} are $g_{00} = g_{11} = g_{22} = g_{33} = -g_{44} = -g_{55} = -g_{66} = 1$.

All linearly independent products of $\hat{\Gamma}_K$ have the same form (5) as for Γ_μ and form again a basis of algebra $gl(8, \mathbb{R})$.

4 Extended SUSY for Aharonov–Bohm–Coulomb and Aharonov–Casher systems

Let us search for extended SUSY of the system defined by the Dirac equation (7) with an external field being a superposition of the Coulomb potential and the potential generated by a solenoid directed along the third co-ordinate axis. Such configuration corresponds to the so-called Aharonov–Bohm–Coulomb (ABC) system which has been studied by a number of investigators (see, e.g., [9, 10]). The related vector-potential has the form

$$eA_0 = \frac{\alpha}{|x|}, \qquad eA_1 = \xi\frac{x_2}{r^2}, \qquad eA_2 = -\xi\frac{x_1}{r^2}, \qquad A_3 = 0, \tag{12}$$

where $r^2 = x_1^2 + x_2^2$.

Using the fact that A_1 and A_2 are locally pure gauges we can prove that there exist constants of motion for the ABC system which are analogues of Johnson–Lippman and Dirac constants of motion for the Coulomb system. They have the following form

$$\begin{aligned} \hat{Q}' &= m\alpha\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{\boldsymbol{x}^2} + iD'\left(\boldsymbol{\sigma}\cdot\boldsymbol{p} + i\gamma_4\frac{\alpha}{|x|}\right), \\ D' &= \gamma_0\left(\boldsymbol{\sigma}\cdot\boldsymbol{J} + \frac{1}{2} + \frac{\xi}{x^2}\left(\sigma_3\boldsymbol{x}^2 - x_3\boldsymbol{\sigma}\cdot\boldsymbol{x}\right)\right) \end{aligned} \tag{13}$$

and commute with the corresponding Hamiltonian

$$H' = \gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{\pi} + \gamma_0 m + \frac{\alpha}{|x|}.$$

Commutation and anticommutation relations for operators $\hat{Q}'$, D' and H' are the same as for unprimed operators considered in the previous section. Thus we can construct two supercharges $Q = \frac{1}{\kappa\sqrt{2}}(1+P)\hat{Q}'$, $\bar{Q} = \frac{1}{\kappa\sqrt{2}}(1-P)\hat{Q}'$ and central charge $Z_{ab} = 2\delta_{ab}(\alpha^2 - D'^2)m^2/\kappa^2$ which satisfy relations (3) together with $\hat{H}' = H'^2$. Thus *the ABC system admits extended SUSY with one non-trivial central charge.*

Additional involutive symmetries for this system can be found in the form

$$\begin{aligned} &R_{12} = i\gamma_1\gamma_2\hat{\theta}_1\hat{\theta}_2, \qquad R_{31} = i\gamma_3\gamma_1\hat{\theta}_3\hat{\theta}_1, \qquad R = \gamma_4\gamma_0\hat{\theta}, \\ &R_{23} = i\exp(i\varphi)\gamma_2\gamma_3\hat{\theta}_2\hat{\theta}_3, \qquad \hat{C} = i\exp(i\varphi)\gamma_2 c. \end{aligned} \tag{14}$$

Here $\varphi = 2\arctan\frac{x_1}{x_2}$, $\hat{\theta}$, $\hat{\theta}_a$ and c are reflection and complex conjugation operators defined in the previous section.

Operators (14) commute with the Dirac operator L of equation (7), with potentials (12) and satisfy the following relations

$$\begin{aligned}&\{R,\hat{Q}'\}=\{\hat{C},Q'\}=\{R,\hat{C}\}=\{\hat{C},R_{ab}\}=\{R_{ab},R_{cd}\}=0,\\&[R_{ab},R]=[R_{ab},Q']=0.\end{aligned}\tag{15}$$

Using (15) we can construct six supercharges for the ABC system, namely

$$\begin{aligned}&Q_1'=(1+\hat{C}R_{12})Q', &&Q_2'=(\hat{C}R_{23}+R)Q', &&Q_3'=\hat{C}(1+R_{31}),\\&\bar{Q}_1'=(1-\hat{C}R_{12})Q', &&\bar{Q}_2'=(\hat{C}R_{23}-R)Q', &&\hat{Q}_3'=\hat{C}(1+R_{31})Q'.\end{aligned}\tag{16}$$

Operators (16) and $\hat{H}'=Q'^2$ satisfy relations (3) and form a basis of $N=6$ extended superalgebra for ABC system. This system admits also the 64-dimensional algebra $gl(8,\mathbb{R})$ of involutive symmetries. Basis elements of this algebra can be obtained using formulae (11) with Q_a', $\bar{Q}_a'$ (16) instead of operators (10).

Let us consider now the relativistic Aharonov–Casher (AC) system [9, 12]. This system includes chargeless particle with non-trivial electric quadrupole momentum, interacting with an infinite homogeneously charged cylinder. It is described by the Dirac equation with anomalous interaction instead of a minimal one:

$$\left(\gamma_\mu p^\mu - m + \frac{ik}{m}\gamma_\mu\gamma_\nu F^{\mu\nu}\right)\psi=0,\tag{17}$$

where $F^{\mu\nu}$ is the strength tensor of the external electromagnetic field generated by infinite homogeneously charged cylinder which we suppose be directed along the third co-ordinate axis.

We shall consider more general system (17) with an external field of the following form

$$F_{ab}=0,\qquad F_{0a}=\frac{\partial\varphi}{\partial x_a},\qquad a,b=1,2,3,$$

where $\varphi=\varphi(\boldsymbol{x})$ is a potential of the electric field which is an even function of spatial variables. In the case $\varphi=\sqrt{x_1^2+x_2^2}$ equation (17) reduces to the AC system.

The considered system admits $N=6$ extended SUSY generated by the following supercharges

$$\begin{aligned}&Q_1=(\Gamma_1+\Gamma_0)H, &&\bar{Q}_1=(\Gamma_1-\Gamma_0)H,\\&Q_2=(\Gamma_2+\Gamma_5)H, &&\bar{Q}_2=(\Gamma_2-\Gamma_5)H,\\&Q_3=(\Gamma_3+\Gamma_6)H, &&\bar{Q}_3=(\Gamma_3-\Gamma_6)H,\end{aligned}\tag{18}$$

where $\Gamma_1,\ldots,\Gamma_6$ are discrete symmetries (6) and

$$H=\gamma_0\gamma_\alpha p^\alpha+\frac{ik}{m}\gamma_\alpha E^\alpha+\gamma_0 m.$$

Operators (18) and $\hat{H}=H^2$ satisfy relations (3) with $Z_{ab}\equiv 0$ and so generate $N=6$ SUSY algebra for the AC system.

The AC system admits also the algebra $gl(8,\mathbb{R})$ whose basis elements are given by relations (6), and so has the same involutive symmetry algebra as the free Dirac equation.

5 Stueckelberg systems

Relativistic Stueckelberg equation [13] describes quantum mechanical systems which have two spin states corresponding to values of spin $s=1$ and $s=0$. It is a system of equations for an antisymmetric tensor field $\psi^{\mu\nu}$, a four-vector field ψ^μ and a scalar field ψ of the following form

$$\begin{aligned}
&p^\mu\psi^\nu - p^\nu\psi^\mu = m\psi^{\mu\nu},\\
&p_\nu\psi^{\mu\nu} = p^\mu\psi + m\psi^\mu,\\
&p_\nu\psi^\nu = m\psi.
\end{aligned} \tag{19}$$

Introducing the minimal and anomalous interaction with an external e.m. field into (19) we obtain the following system

$$\begin{aligned}
&\pi^\mu\psi^\nu - \pi^\nu\psi^\mu = m\psi^{\mu\nu},\\
&\pi_\nu\psi^{\mu\nu} = \pi^\mu\psi + m\psi^\mu + \frac{e}{m}F_{\mu\nu}\psi^\nu,\\
&\pi_\nu\psi^\nu = m\psi.
\end{aligned} \tag{20}$$

Here $\pi_\mu = p_\mu - eA_\mu$ and $F_{\mu\nu} = -\frac{i}{e}[\pi_\mu, \pi_\nu]$ is the strength tensor of the electromagnetic field.

A special form of anomalous interaction chosen in (20) yields to extended SUSY for this equation. Other interactions for the Stueckelberg equation are discussed in [14].

Expressing $\psi^{\mu\nu}$, and ψ in (20) via ψ^μ we come to the second-order equation

$$\left(\pi_\nu\pi^\nu - m^2\right)\psi^\mu + 2eF^{\mu\nu}\psi_\nu = 0. \tag{21}$$

Its symmetries will be investigated in few steps.

We begin with the constant and homogeneous external magnetic field directed along the third co-ordinate axis. The corresponding vector-potential and tensor $F^{\mu\nu}$ have the form

$$A_0 = A_2 = A_3 = 0, \qquad A_1 = -Hx_2, \qquad F_{0a} = F_{23} = F_{31} = 0, \qquad F_{12} = H. \tag{22}$$

Substituting (22) into (21) and representing ψ_ν as

$$\psi_\nu = \exp\left(iEt + ip_1x_1 + ip_3x_3\right)\varphi_\nu(x_2), \qquad x_2 = \left(\frac{p_1}{\sqrt{eH}} + y\right)$$

equation (21) can be reduced to the form

$$E^2\varphi = \left(m^2 + p_3^2 - \frac{\partial^2}{\partial y^2} + \omega^2y^2 + 2S_3\omega\right)\varphi, \tag{23}$$

where $\omega = eH$, $\varphi = \text{column}(\varphi_0\varphi_1\varphi_2\varphi_3)$ and

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Equation (23) admits a large extended supersymmetry. First, we indicate two sets of constants of motion which generate extended SUSY with non-trivial central charges. The basis elements of the corresponding superalgebras have the following forms:

$$\begin{aligned}
&\tilde{Q}_1 = \frac{1}{2}(\sigma_1 + i\sigma_2)(p + i\omega y), \qquad \bar{\tilde{Q}}_1 = \frac{1}{2}(\sigma_1 - i\sigma_2)(p - i\omega y),\\
&\hat{H} = -\frac{\partial^2}{\partial y^2} + \omega^2y^2 + 2S_3\omega + p_3^2 + m^2, \qquad \tilde{Z}_{ab} = 2\delta_{ab}\left(p_3^2 + m^2 - \frac{1}{2}\tau_3\omega\right)
\end{aligned} \tag{24}$$

and

$$\tilde{Q}_1' = \frac{1}{2}(\tau_1 + i\tau_2)(p + i\omega y), \qquad \bar{\tilde{Q}}_1' = \frac{1}{2}(\tau_1 - i\tau_2)(p - i\omega y),$$
$$\hat{H}' = \hat{H}, \qquad \tilde{Z}_{ab}' = 2\delta_{ab}\left(p_3^2 + m^2 - \frac{1}{2}\sigma_3\omega\right). \tag{25}$$

Here

$$\sigma_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\tau_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

It can be verified by a direct calculation that operators (24) and (25) commute with $\hat{H}$ and are, therefore, constants of motion for equation (23). In addition, operators (24) and (25) satisfy relations (3), so *the Stueckelberg equation with the constant and homogeneous external magnetic field admits extended SUSY with non-trivial central charges.*

The superalgebras (24) and (25) can be jointed in frames of a more extended superalgebra including ten elements. Four of them are odd elements (supercharges), namely, Q_α, $\bar{Q}_\alpha$, $\alpha = 1, 2$:

$$Q_1 = \frac{1}{2}\left(\sigma_1 p + \sigma_2\omega y + i\sigma_3(\tau_1 p + \tau_2\omega y)\right),$$
$$\bar{Q}_1 = Q_1^\dagger = \frac{1}{2}\left(\sigma_1 p + \sigma_2\omega y - i\sigma_3(\tau_1 p + \tau_2\omega y)\right),$$
$$Q_2 = \frac{1}{2}(\sigma_2 p - \sigma_1\omega y + i\sigma_3(\tau_2 p - \tau_1\omega y)),$$
$$\bar{Q}_2 = Q_2^\dagger = \frac{1}{2}(\sigma_2 p - \sigma_1\omega y - i\sigma_3(\tau_2 p - \tau_1\omega y)) \tag{27}$$

and six of them are even. They include the central charge $Z_{ab} = 2\delta_{ab}(p_3^2 + m^2)$, and five additional elements of the form

$$\hat{H} = -\frac{\partial^2}{\partial y^2} + \omega^2 y^2 + 2S_3\omega + p_3^2 + m^2, \qquad I_0 = \omega(\sigma_3 + \tau_3)/2,$$
$$I_1 = \omega(\sigma_2\tau_1 - \sigma_1\tau_2)/2, \qquad I_\pm = \omega\left(\sigma_3 - \tau_3 \pm (\sigma_1\tau_1 + \sigma_2\tau_2)\right)/4. \tag{28}$$

Anticommutation relations for odd elements are given by the following formulae

$$\{Q_a, Q_b^\dagger\} = \delta_{ab}(H - Z - I_0) - i\varepsilon_{ab}I_1, \qquad \{Q_a, Q_b\} = \delta_{ab}I_-, \tag{29}$$

where $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$. The remaining (commutation) relations of odd elements with even and even elements with even ones are of the form

$$[Q_a, \hat{H}] = [I_0, \hat{H}] = [I_\pm, \hat{H}] = 0, \qquad [Q_a, I_0] = -i\varepsilon_{ab}Q_b, \qquad [Q_a, I_1] = Q_a,$$
$$[Q_a, I_-] = 0, \qquad [Q_a, I_+] = -i\varepsilon_{ab}Q_b^\dagger, \qquad [I_0, I_1] = [I_0, I_\pm] = 0,$$
$$[I_1, I_\pm] = \pm I_\pm, \qquad [I_+, I_-] = I_1. \tag{30}$$

In addition Z commutes with all operators enumerated in (27), (28).

Thus Stueckelberg particle interacting with a constant homogeneous external magnetic field forms a system admitting extended superalgebra characterized by relations (29), (5). We will further denote this algebra as $\mathcal{A}$.

Consider now Stueckelberg equation for the case when external field is generated by a point charge. The related vector-potential can be chosen in the form (8) and equation (21) reads

$$\left(p_0 + \frac{\alpha^2}{|x|}\right)^2 \Psi = \left(\boldsymbol{p}^2 + m^2 + i\alpha\frac{(\sigma_a - \tau_a)x_a}{|x|^3}\right)\Psi, \tag{31}$$

where $\Psi = \text{column}\,(\psi_0, \psi_1, \psi_2, \psi_3)$.

Rather surprisingly, equation (31) also admits extended invariance superalgebra, isomorphic to $\mathcal{A}$. This can be shown by writing Ψ in the form $\Psi = \exp(iEt)\varphi(\boldsymbol{x})$ (i.e., considering the related eigenvalue problem) and introducing new space variables $\boldsymbol{r} = E\boldsymbol{x}$. Equation (31) then takes the form

$$\mu\varphi = \hat{H}\varphi,$$

where

$$\hat{H} = p'^2 + i\alpha\frac{(\sigma_a - \tau_a)r_a}{|r|^3} - \left(\frac{\alpha}{|r|} - 1\right)^2, \qquad \text{and} \qquad \mu = -\frac{m^2}{E^2}.$$

The corresponding radial equation can be written as [15]:

$$\mu\varphi(r) = \hat{H}\varphi \equiv \left(-\frac{d^2}{dr^2} + \begin{pmatrix} V_1 & 0 & 0 & 0 \\ 0 & V_2 & 0 & 0 \\ 0 & 0 & V_3 & 0 \\ 0 & 0 & 0 & V_4 \end{pmatrix}\right)\varphi(r), \tag{32}$$

where

$$V_1 = \frac{b^2 - \frac{1}{4}}{r^2} - \frac{2\alpha}{r}, \qquad V_2 = V_3 = \frac{(b+1)^2 - \frac{1}{4}}{r^2} - \frac{2\alpha}{x}, \qquad V_4 = \frac{(b-1)^2 - \frac{1}{4}}{r^2} - \frac{2\alpha}{r}. \tag{33}$$

It can be proven by a direct verification that equation (32) admits nine constants of motion, namely

$$Q_1 = \begin{pmatrix} 0 & a_-^1 & ia_-^1 & 0 \\ a_+^1 & 0 & 0 & -ia_-^2 \\ ia_+^1 & 0 & 0 & a_-^2 \\ 0 & -ia_+^2 & a_+^2 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & -ia_-^1 & a_-^1 & 0 \\ ia_+^1 & 0 & 0 & -a_-^2 \\ -a_+^1 & 0 & 0 & -ia_-^2 \\ 0 & a_+^2 & ia_+^2 & 0 \end{pmatrix},$$

$$\bar{Q}_1 = \begin{pmatrix} 0 & a_+^1 & -ia_+^1 & 0 \\ a_-^1 & 0 & 0 & +ia_+^2 \\ -ia_-^1 & 0 & 0 & a_+^2 \\ 0 & ia_+^2 & a_-^2 & 0 \end{pmatrix}, \qquad \bar{Q}_2 = \begin{pmatrix} 0 & +ia_+^1 & -a_+^1 & 0 \\ -ia_-^1 & 0 & 0 & a_+^2 \\ a_-^1 & 0 & 0 & ia_+^2 \\ 0 & -a_-^2 & -ia_+^2 & 0 \end{pmatrix},$$

$$I_0 = C\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad I_1 = iC\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$I_\pm = C\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Z = m^2 + p_3^2 \tag{34}$$

which satisfy relations (29) and (5). Here

$$a_{\pm}^{1} = p' \pm i\left(\frac{b+\frac{1}{2}}{x} + \frac{\alpha}{b+\frac{1}{2}}\right), \qquad a_{\pm}^{2} = p' \pm i\left(\frac{b-\frac{1}{2}}{x} + \frac{\alpha}{b-\frac{1}{2}}\right),$$
$$C = \frac{\alpha b}{a^2-\frac{1}{4}}, \qquad b^2 = \left(j+\frac{1}{2}\right)^2$$

and j is the quantum number defining the spectrum of total angular momentum $\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{p}' + \boldsymbol{\sigma} + \boldsymbol{\tau}$ of a system in state Ψ, i.e., $\boldsymbol{J}^2\Psi = j(j+1)\Psi$.

Thus the Stueckelberg equation with Coulomb potential is invariant with respect to extended superalgebra whose generators are given in (34). As a consequence it admits two symmetry superalgebras (24) and (25), and hence is characterized by extended SUSY with non-trivial central charges. Using discrete involutive symmetries of the Stueckelberg equation it is possible to construct extra supercharges which enlarge superalgebras (24) and (25) to $N = 6$ extended SUSY.

6 Representations of superalgebra $\mathcal{A}$

In order to describe other QM systems invariant with respect to superalgebra $\mathcal{A}$ which admit extended SUSY we construct representations of this algebra realized by differential operators defined on four-component vector-functions. The related supercharges and even elements of the superalgebra can then chosen in the form (34) where

$$a_1^{\pm} = p \pm iW_1, \qquad a_2^{\pm} = p \pm iW_2, \qquad p = -i\frac{d}{dx}. \tag{35}$$

Here W_1 and W_2 are functions of x satisfying the following relation

$$W_1^2 - W_2^2 + W_1' + W_2' = C \tag{36}$$

with C being a constant and prime denoting derivative of W_α with respect to x.

Operators (34), (35) satisfy relations (29), (5) for the case when W_1, W_2 are arbitrary functions satisfying condition (36). Choosing $W_1 = W_2 = \omega x$ in (35) we obtain supercharges for the Stueckelberg system with constant, homogeneous external magnetic field. The choice $W_1 = \frac{b+\frac{1}{2}}{x} + \frac{\alpha}{b+\frac{1}{2}}$, $W_2 = \frac{b-\frac{1}{2}}{x} + \frac{\alpha}{b-\frac{1}{2}}$ corresponds to the Stueckelberg–Coulomb system. Two other choices, namely, $W_1 = -W_2 = \omega x$ and $W_1 = -W_2 = \frac{b}{x} + \frac{\alpha}{b}$ correspond to Dirac particle in the constant, homogeneous external magnetic field and Coulomb field respectively, where all states have additional two fold degeneracy.

7 Discussion

We have shown that extended SUSY with non-trivial central charges appears as internal symmetry of many quantum mechanical systems. In particular we have proven that symmetry of the relativistic Coulomb system as well as of the Aharonov–Bohm–Coulomb and Aharonov–Casher systems can be described by the superalgebra including six supercharges. The Stueckelberg systems systems are characterized even by more extended SUSY described by ten-dimensional superalgebra with non-trivial central charges.

One more goal of our analysis was searching for realistic quantum mechanical systems which are invariant with respect to algebra $gl(8,\mathbb{R})$ of involutive discrete symmetries. This invariance algebra for the free Dirac equation was found in papers [3, 6]. In the present paper we prove that this symmetry is valid also for the Coulomb, ABC and AC systems.

A natural question arises what are the practical consequences of the found symmetries. Using the technique developed in [3, 7] it is possible to use $gl(8,\mathbb{R})$ symmetry to decouple the related Dirac equation and construct complete sets of solutions.

A standard application of SUSY consists in prediction and interpretation of degeneration of energy spectra of the related QM systems. Energy levels for the exactly solvable Coulomb–Dirac problem are degenerated with respect to quantum numbers sign j_3 and sign κ where j_3 and κ are eigenvalues of mutually commuting operators of the third component of the total angular momentum J_3 and D respectively. One more degeneration which is non-observable is connected with the change of sign of the phase multiplier of the Dirac-Coulomb wave function. Extended SUSY presents a specific interpretation of these degenerations. A particular importance of such interpretation consists in the fact that such a degeneration appears for all the systems which admit extended SUSY, e.g., for the AC system.

The other application of (extended) SUSY is to construct exact solutions of QM systems with sharp invariant potentials using purely algebraic methods [2] which admit a straitforward generalization to the case of a more general superalgebra (29), (5). The potentials (33) of the Stueckelberg–Coulomb system are shape invariant which enables us to find easily its energy eigenvalues. They can be written in the following form

$$E_{n\kappa\lambda} = m\left[1+\frac{\alpha^2}{\left(n+\frac{1}{2}+b+\lambda\right)^2}\right]^{\frac{1}{2}},$$

where $n = 0, 1, 2, \ldots$, $\lambda = 0, \pm 1$, $b = \sqrt{\kappa^2 - \alpha^2}$, $|\kappa| = 1, 2, \ldots$

Finally we notice that representations of superalgebra $\mathcal{A}$ considered in the previous section can be used in non-relativistic quantum mechanics. It seems to us that such generalized SUSY quantum mechanics has better physical grounds than parasupersymmetric quantum mechanics [16, 17] and $n = N$ $(N > 1)$ SUSY quantum mechanics [18], since it is realized in a number of quite realistic QM systems. We plane to study possible applications of superalgebra in quantum mechanics $\mathcal{A}$ elsewhere.

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[1] Witten E., Dynamical breaking of supersymmetry, *Nucl. Phys. B*, 1981, V.188, 513–554.

[2] Cooper F., Khare A. and Sukhatme U., Supersymmetry and quantum mechanics, *Phys. Rep.*, 1995, V.251, 267–385.

[3] Niederle J. and Nikitin A.G., Involutive symmetries, supersymmetries and reductions of the Dirac equation, *J. Phys. A*, 1997, V.30, 999–1010.

[4] Tkachuk V.M. and Vakarchuk S.I., Supersymmetry of the electron in a three-dimensional magnetic field, *Phys. Lett. A*, 1997, V.228, N 3, 141–145.

[5] Niederle J. and Nikitin A.G., Extended supersymmetries for Schrödinger–Pauli equation, *J. Math. Phys.*, 1999, V.40, N 3, 1280–1293.

[6] Niederle J. and Nikitin A.G., Non-Lie and discrete symmetries of the Dirac equation, *J. Nonlin. Math. Phys.*, 1997, V.4, 436–444.

[7] Nikitin A.G., Algebras of discrete symmetries and sypersymmetries for the Schrödinger–Pauli equation, *Int. J. of Mod. Phys.*, 1998, V.14, N 6, 885–897.

[8] Johnson M.P. and Lippmann B.A., Electromagnetic signals from nuclear explosions in Outer space, *Phys. Rev.*, 1960, V.119, 827–828.

[9] Hagen C.R., Effects of nongauge potentials on the spin 1/2 Aharonov–Bohm problem, *Phys. Rev. D*, 1993, V.48, 5935–5939.

[10] Park D.L. and Yoo S.-K, Propagator for spinless and spin 1/2 Aharonov–Bohm Coulomb systems, *Ann. Phys.*, 1998, V.263, 295–309.

[11] Hagen C.R., Exact equivalence of spin-$\frac{1}{2}$ Aharonov–Bohm and Aharonov–Casher effects, *Phys. Rev. Lett.*, 1990, V.64, N 20, 2347–2349.

[12] Bruce S., Roa L., Saavedra C. and Klimov A.B., Unbroken supersymmetry in the Aharonov–Casher effect, *Phys. Rev. A*, 1999, V.60, N 1, R1–R4.

[13] Stueckelberg E., Die Wechelvirkungckrafte in der Electrodynamic und der Feldtheorie der Kernkraefle, *Helv. Phys. Acta*, 1938, V.11, 225–236.

[14] Beckers J., Debergh N. and Nikitin A.G., On parasupersymmetries and relativistic description for spin one particles: I. The free particle context, *Fortschritte der Physik*, 1995, V.43, N 1, 67–80; II. The interacting context with (electro)magnetic fields, *Fortschritte der Physik*, 1995, V.43, N 1, 81–95.

[15] Fushchich W.I. and Nikitin A.G., Symmetries of equations of quantum mechanics, New York, Allerton Press, 1994.

[16] Rubakov V.A. and Spiridonov V.P., Parasupersymmetric quantum mechanics, *Mod. Phys. Lett.*, 1988, V.3, N 14, 1337–1347.

[17] Beckers J. and Debergh N., Parastatistics and supersymmetry in quantum mechanics, *Nucl. Phys. B*, 1990, V.340, N 2–3, 767–776.

[18] Donets E.E., Pashnev A., Rosales J.J. and Tsulaia M.M., $N = 4$ supersymmetric multidimensional quantum mechanics, partial SUSY breaking, and superconformal quantum mechanics, *Phys. Rev. D*, 2000, V.61, N 4, 043512.