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On a CFT Prediction in the Sine-Gordon Model

Carlos NAON and Mariano SALVAY

Instituto de Física La Plata, Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, C.C. 67, La Plata 1900, Argentina
E-mail: *naon@fisica.unlp.edu.ar, salvay@fisica.unlp.edu.ar*

A quantitative prediction of Conformal Field Theory (CFT), which relates the second moment of the energy-density correlator away from criticality to the value of the central charge, is verified in the sine-Gordon model. By exploiting the boson-fermion duality of two-dimensional field theories, this result also allows to show the validity of the prediction in the strong coupling regime of the Thirring model.

Some time ago Cardy [1] derived a quantitative prediction of conformal invariance [2, 3] for 2D systems in the scaling regime, away from the critical point. Starting from the so called 'c-theorem' [4], he was able to relate the value of the conformal anomaly c, which characterizes the model at the critical point, to the second moment of the energy-density correlator in the non-critical theory:

$$\int d^2x\,|x|^2\,\langle\varepsilon(x)\varepsilon(0)\rangle = \frac{c}{3\,\pi\,t^2\,(2-\Delta_\varepsilon)^2}, \tag{1}$$

where ε is the energy-density operator, Δ_ε is its scaling dimension and $t \propto (T-T_c)$ is the coupling constant of the interaction term that takes the system away from criticality. It is interesting to notice that a similar sum rule has been recently obtained by Jancovici [5] in the context of Classical Statistical Mechanics. This author considered the correlations of the number-density of particles in a 2D two-component plasma (Coulomb gas).

The validity of (1) has been explicitly verified for the Ising model [1], for 2D self-avoiding rings [6], and for the Baxter model [7]. In this last case the formula could be checked only in the weak-coupling limit, by describing the system in terms of a massive Thirring model [8] and performing a first-order perturbative computation.

The main purpose of this note is to show that (1) also holds for a bosonic QFT with highly non-trivial interactions, the well-known sine-Gordon (SG) model with Euclidean Lagrangian density given by

$$L = \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{\alpha}{\lambda}\cos\left(\sqrt{\lambda}\Phi\right) + \frac{\alpha}{\lambda}, \tag{2}$$

where α and λ are real constants.

In this work we shall perform a perturbative computation up to second order in λ. We are then naturally led to consider the renormalization of this theory. Fortunately this issue has been already analyzed by many authors [9–14]. One of the main conclusions is that for $\lambda < 8\pi$ a normal order procedure that eliminates the contributions of the tadpoles is enough to have a finite theory. The only effect of this prescription is to renormalize the constant α. We shall then restrict our study to this case.

Since we want to verify equation (1) for the model given by (2), we will take $\varepsilon = (1 - \cos\sqrt{\lambda}\Phi)/\lambda$ and $t = \alpha$. Taking into account that the model of free massless scalars has a conformal charge $c = 1$ and that the scaling dimension of ε for this case is equal to $\lambda/4\pi$, (1) reads

$$F(\alpha,\lambda) = \int d^2x\,|x|^2\,\langle\varepsilon(x)\varepsilon(0)\rangle = \frac{1}{3\,\pi\,\alpha^2\left(2-\frac{\lambda}{4\pi}\right)^2}. \tag{3}$$

Expanding the interaction term up to order λ^2 one has $\varepsilon = \frac{1}{2}\Phi^2 - \frac{\lambda}{4!}\Phi^4 + \frac{\lambda^2}{6!}\Phi^6$. Replacing this expression in (3) we obtain

$$F(\alpha,\lambda) = A(\alpha,\lambda) + B(\alpha,\lambda) + C(\alpha,\lambda) + D(\alpha,\lambda),$$

where

$$A(\alpha,\lambda) = \frac{1}{4}\int d^2x\,|x|^2\,\langle\Phi^2(x)\Phi^2(0)\rangle_\alpha, \qquad B(\alpha,\lambda) = -\frac{\lambda}{4!}\int d^2x\,|x|^2\,\langle\Phi^2(x)\Phi^4(0)\rangle_\alpha,$$
$$C(\alpha,\lambda) = \frac{\lambda^2}{(4!)^2}\int d^2x\,|x|^2\,\langle\Phi^4(0)\Phi^4(x)\rangle_\alpha, \qquad D(\alpha,\lambda) = \frac{\lambda^2}{6!}\int d^2x\,|x|^2\,\langle\Phi^2(x)\Phi^6(0)\rangle_\alpha.$$

At this point we notice that $D(\alpha,\lambda)$, up to this order, contains only tadpoles which, as explained above, were already considered in the renormalization of α. Then we must disregard this contribution in the present context. Now, in order to illustrate the main features of the computation, we shall briefly describe the evaluation of $A(\alpha,\lambda)$, which involves both analytical and numerical procedures. In the above equations $\langle\ \rangle_\alpha$ means v.e.v. with respect to the SG Lagrangian expanded up to second order in λ. From now on we will decompose this Lagrangian into free and interaction pieces as

$$L_0 = \frac{1}{2}(\partial_\mu\Phi)^2 + \frac{\alpha}{2}\,\Phi^2, \qquad L_{\text{int}} = -\frac{\alpha\lambda}{4!}\,\Phi^4 + \frac{\alpha\lambda^2}{6!}\,\Phi^6.$$

Using Wick's theorem and the well-known expression for the free bosonic propagator $\langle\Phi(x)\Phi(0)\rangle_0 = 1/(2\pi)K_0(\sqrt{\alpha}|x|)$ (K_0 is a modified Bessel function of zeroth order), after a convenient rescaling of the form $x \to \frac{x}{\sqrt{\alpha}}$, we obtain

$$\begin{aligned}A(\alpha,\lambda) = {} & \frac{1}{2\alpha^2(2\pi)^2}\int d^2x\,|x|^2\,K_0^2(|x|) + \frac{\lambda}{4\alpha^2(2\pi)^4}\iint d^2x\,d^2x_1\,|x|^2\,K_0^2(|x_1-x|)K_0^2(|x_1|)\\ & + \frac{\lambda^2}{\alpha^2(2\pi)^6}\iiint d^2x\,d^2x_1\,d^2x_2\,|x|^2\Bigg[\frac{1}{8}K_0^2(|x_2-x_1|)K_0^2(|x_1|)K_0^2(|x_2-x|)\\ & + \frac{1}{6}K_0^3(|x_2-x_1|)K_0(|x_2|)K_0(|x_1-x|)K_0(|x|)\\ & + \frac{1}{4}K_0^2(|x_2-x_1|)K_0(|x_1|)K_0(|x_2-x|)K_0(|x_2|)K_0(|x_1-x|)\Bigg].\end{aligned}$$

The first two terms are related to tabulated integrals [15] yielding the first order analytical expression $F(\alpha,\lambda) = (1+\lambda/(4\pi))/\left(12\pi\alpha^2\right)$ which can be shown to verify (3) in a straightforward way. Concerning the remaining second order terms we have three contributions corresponding to the prefactors 1/8, 1/6 and 1/4 in the last integrand. Due to its symmetry, the first one can be analytically computed by repeatedly using

$$\int d^2x\,|x|^2\,K_0^2(|x-y|) = \frac{2\pi}{3} + \pi|y|^2.$$

The result is $2\pi^3$. The computation of the other two contributions is more involved. In fact we could not find analytical results in these cases. However we were able to considerably simplify these multiple integrals in order to facilitate their numerical evaluation. Indeed, using the Fourier transform of K_0 and the integral representation of the first kind Bessel function J_0:

$$J_0(|x|) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\,\exp\left(-i|x|\cos\theta\right),$$

we obtain

$$\iiint d^2x\, d^2x_1\, d^2x_2\, |x|^2\, K_0^3(|x_2 - x_1|) K_0(|x_1|) K_0(|x_2 - x|) K_0(|x|)$$
$$= 4(2\pi)^3 \int dr \int dp\, \frac{r\, p\, \left(1 - p^2\right) K_0^3(r) J_0(pr)}{\left(p^2 + 1\right)^5}.$$

Using NIntegrate in the program Mathematica for the double integral in this expression one obtains the value 0.04874. Similar manipulations with the last contribution to $A(\alpha, \lambda)$ led us to the following numerical results:

$$\int dr \int dp \int dk \frac{r\, p\, k\, \left(1 - p^2\right) K_0^3(r) J_0(pr) J_0(kr)}{\left(p^2 + 1\right)^4 \left(k^2 + 1\right)^2} = 0.0169622,$$

and

$$\int dr\, r^5\, K_0^2(r) K_1^2(r) = 0.08783.$$

Putting all this together we have

$$A(\alpha, \lambda) = \frac{1}{12\pi\alpha^2} \left(1 + \frac{\lambda}{4\pi}\right) + \frac{\lambda^2}{\alpha^2 (2\pi)^6} \left[\frac{1}{8}\, 2\pi^3 + \frac{1}{6}\, 4\, (2\pi)^3\, 0.04874\right.$$
$$\left. + \frac{1}{4}\, (2\pi)^3 \left(8 \times 0.0169622 - \frac{1}{8}\, 0.08783\right)\right].$$

Working along the same lines with $B(\alpha, \lambda)$ and $C(\alpha, \lambda)$ we find

$$B(\alpha, \lambda) = \frac{-32\lambda^2}{\alpha^2 (2\pi)^3 4!} \int dr \int dp\, \frac{r\, p\, \left(1 - p^2\right) K_0^3(r) J_0(p\, r)}{\left(p^2 + 1\right)^4} = \frac{-32\lambda^2}{\alpha^2 (2\pi)^3 4!} \times 0.0501125$$

and

$$C(\alpha, \lambda) = \frac{\lambda^2}{\alpha^2 (2\pi)^3 4!} \int dr\, r^3\, K_0^4(r) = \frac{\lambda^2}{\alpha^2 (2\pi)^3 4!} \times 0.0754499.$$

All these numerical values were confirmed by using Fortran.

Finally, inserting these results in the left hand side of (3) we get

$$F(\alpha, \lambda) = \frac{1}{12\pi\alpha^2} + \frac{\lambda}{48\pi^2\alpha^2} + \frac{\lambda^2}{4!(2\pi)^3\alpha^2} \left(\frac{3}{4} + 5.4 \times 10^{-6}\right).$$

Comparing this expression with the expansion in λ of the right hand side of (3) one sees that they are equal up to first order and differ in a small quantity $\left(O\left(10^{-6}\right)\right)$ up to second order in λ.

In summary, we have verified the validity of a quantitative prediction of CFT in the context of the SG model. Taking into account the well-known bosonization identity between the SG theory and the massive Thirring model (characterized by a coupling g^2) [9] which takes place for $\beta^2/(4\pi) = \left(1 + g^2/\pi\right)^{-1}$, it becomes apparent that our result implies that (1) also holds in the strong coupling limit of the Thirring model. This, in turn, allows us to improve the proof of the validity of (1) for the Baxter and Ashkin–Teller models which, up to now, was restricted to the weak coupling limit [7].

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