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New Exact Solutions of Khokhlov–Zabolotskaya–Kuznetsov Equation

Paulius MIŠKINIS

Vilnius Gediminas Technical University, Faculty of Fundamental Sciences, Department of Physics, 11 Saulėtekio Ave, LT-2040 Vilnius, Lithuania
E-mail: *paulius.miskinis@fm.vtu.lt*

Khokhlov–Zabolotskaya–Kuznetsov equation $(\phi_t + \phi\phi_x - \alpha\phi_{xx})_x - 1/2(\phi_{yy} + \phi_{zz}) = 0$ and its solutions are analyzed. A series of complete exact analytical solutions related to the one-dimensional and vectorial inhomogeneous Burgers equation are presented. A concrete example which corresponds to a special form of the inhomogeneous term is analyzed. Reduction to the traveling wave solution is considered.

1 Introduction

The Khokhlov–Zabolotskaya–Kuznetsov equation (KhZKE) describes the evolution of the spreading of nonlinear diffraction waves whose cross-section is large compared to their length. This is one of the basic equations of nonlinear wave processes. As the generalized KhZKE usually the equation

$$\frac{\partial}{\partial \tau}\left(\frac{\partial p}{\partial z} - \lambda p \frac{\partial p}{\partial \tau} + \hat{L}p\right) = \frac{c}{2}\Delta_\perp p \tag{1}$$

is accepted, where $p = p(z,\tau)$ usually means pressure, z, t are space and time coordinates, $\lambda = \varepsilon/c_0^3\rho_0$ is a parameter characterizing nonlinearity, c is the velocity of sound in the medium; $\Delta_\perp = \Delta(x,y)$ is a two-dimensional Laplacian according to the parameters in the cross-section of the wave packet; $\hat{L}$ in the general case is an integro-differential operator determined by the frequency dependence of weak dispersion and dissipative properties of the medium. Most frequently a generalization of KhZKE containing the second derivative

$$\hat{L} = -b\frac{\partial^2}{\partial \tau^2} \tag{2}$$

is used, which describes dissipation, the finite width of the weak shock wave front in particular.

KhZKE (1) looks rather awkward, nevertheless, it is known to have the exact analytical solution [1]. The present work contains the whole series of exact KhZKE solutions with the second-order operator $\hat{L}$. A concrete solution corresponding to the traveling wave solution is considered.

2 One-dimensional case

Let us divide our search for KhZKE solution into two stages. First of all, we will write the KhZKE as an inhomogeneous Burgers equation and then try to find its exact complete solutions. For the sake of further simplification, it is feasible to represent the constant b in the expression for operator $\hat{L}$ as

$$b \to \frac{b}{2c^3\rho}, \tag{3}$$

where ρ is the density index of the medium.

Then, through substitution of variables

$$z \to \frac{1}{\lambda p_0} t, \quad \tau \to -x, \quad p \to p_0 \phi, \quad \frac{b}{2\varepsilon p_0} \to \nu, \quad x \to \frac{2\lambda p_0}{c} x, \quad y \to \frac{2\lambda p_0}{c} y, \tag{4}$$

KhZKE (1) transforms into

$$(\phi_t + \phi\phi_x - \alpha\phi_{xx})_x - \frac{1}{2}(\phi_{yy} + \phi_{zz}) = 0. \tag{5}$$

By integrating equation (5) by x variable, let us represent the KhZKE as an inhomogeneous Burgers equation (IBE):

$$\phi_t + \phi\phi_x - \alpha\phi_{xx} = \beta f, \tag{6}$$

where $f = 1/2 \int_{x_0}^{x} (\phi_{yy} + \phi_{zz})\, dx$, and β is a certain constant introduced to ensure the possibility of changing the influence of the inhomogeneous term $f = f(x, y, z, t)$. In a whole series of cases when the dependence of $\phi(y, z)$ solution is negligible, or if we are interested in the asymptotic solution resulting form the mediumization of the initial equation KhZK (zonal mediumization, Reynold's mediumization, etc.), the righthand part can be presented as $f(x, t)$.

The Hopf and Cole transformation [2, 3]

$$\phi = -2\alpha\partial_x \ln w \tag{7}$$

relates each solution of the diffusion equation (DE)

$$w_t = \alpha w_{xx} \tag{8}$$

to a corresponding solution $\phi(x, t)$ of Burgers equation (BE) [4]:

$$\phi_t + \phi\phi_x - \alpha\phi_{xx} = 0\,. \tag{9}$$

This allows a detail analysis of the formation and evolution of shock waves in a nonlinear environment.

However, upon introducing into equation (8) even a simplest inhomogenous term, the interrelation between BE and DE through Hopf and Cole transformation (7) disappears. DE (8) is a simplest parabolic equation, therefore, searching for solutions of the inhomogeneous diffusion equation

$$w_t + \alpha w_{xx} = h(x, t) \tag{10}$$

and of the corresponding inhomogeneous BE (9) generalization, approximate methods of calculation (most frequently the method of finite differences) are applied [5, 6].

To obtain a pithy inhomogeneous generalization of BE, let us consider a commutative diagram:

$$\begin{array}{ccc} SE & \xrightarrow{t\to -it} & DE \\ {\scriptstyle h^{-1}}\downarrow & & \uparrow{\scriptstyle h} \\ SENT & \xleftarrow[t\leftarrow it]{} & IBE \end{array} \tag{11}$$

where SE is Schrödinger equation, DE is diffusion equation, IBE is inhomogeneous Burgers equation (6) and SENT is Schrödinger equation with a nonlinear term (not to be mixed with nonlinear Schrödinger equation). The map h is the Hopf–Cole transformation (7) and h^{-1} is the inverse Hopf–Cole transformation

$$w \xrightarrow{h^{-1}} w_0 \exp\left\{-\frac{1}{2\alpha}\int \phi(x,t)\,dx\right\}. \tag{12}$$

It is important that IBE (6) can be got by the transformation (7) of a linear type diffusion equation

$$w_t = \alpha w_{xx} - \frac{\beta}{2\alpha}F(x,t)w, \tag{13}$$

where

$$F(x,t) = \int_{x_0}^{x} d\xi\, f(\xi,t) + C(t), \tag{14}$$

with x_0 as an arbitrary constant, while $C(t)$ is an arbitrary function of t.

3 The vectorial Khokhlov–Zabolotskaya–Kuznetsov equation

While studying the spread of nonlinear waves in a three-dimensional space not in one, but in all spatial directions, it is the three-dimensional vectorial Khokhlov–Zabolotskaya–Kuznetsov equation that suits the purpose best:

$$\boldsymbol{\nabla}\left[\boldsymbol{\phi}_t + (\boldsymbol{\phi}\boldsymbol{\nabla})\boldsymbol{\phi} - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi})\right] - \boldsymbol{\nabla}^2\boldsymbol{\phi} = 0, \tag{15}$$

where $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{x},t) \in \mathbb{R}^3$, $\alpha > 0$, and $\boldsymbol{\nabla}$ is the gradient operator. If the influence of the medium from the right-hand side of the equation can be reduced effectively to a function on space and time coordinates, then the corresponding vectorial inhomogeneous Burgers equation (VIBE) is

$$\boldsymbol{\phi}_t + (\boldsymbol{\phi}\boldsymbol{\nabla})\boldsymbol{\phi} - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi}) = \beta\boldsymbol{f}, \tag{16}$$

where $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},t)$ is a function only of space–time coordinates.

In hydrodynamics, the VIBE and VBE are obtained by refusing the condition that the pressure gradient disappears in the direction perpendicular to the direction of motion of the nonlinear wave: $\boldsymbol{\nabla}_{\perp}p = 0$ [7]. Such equation, together with the continuity equation, was proposed to study the cosmological models of the Early Universe [8, 9]. Only comparatively recently a mathematically strict notion of the generalized solution of such a system was suggested, and it shows that the variational representation of the generalized solution in the two-dimensional case essentially differs from that of the one-dimensional case [10].

Like in the one-dimensional case, the linear type diffusion equation

$$w_t = \boldsymbol{\nabla}(\boldsymbol{\nabla}w) - \frac{\beta}{2\alpha}F(\boldsymbol{r},t)w, \tag{17}$$

by the vectorial generalization of Hopf–Cole transformation

$$\boldsymbol{\phi}(\boldsymbol{r},t) = -2\alpha\boldsymbol{\nabla}\ln w \tag{18}$$

can be mapped into an VIBE (16), where

$$F(\boldsymbol{r},t) = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} d\boldsymbol{\xi}\, \boldsymbol{f}(\boldsymbol{\xi},t) + C(t), \tag{19}$$

with $\boldsymbol{r}_0$ as an arbitrary constant, while $C(t)$ is an arbitrary function of t.

The solution of linear equation (17) is

$$w(\boldsymbol{r},t) = \int d\boldsymbol{r'}\, K(\boldsymbol{r},t,\boldsymbol{r'},0)\, w(\boldsymbol{r'},0), \tag{20}$$

where the kernel $K(\boldsymbol{r},t,\boldsymbol{r'},0)$ satisfies the heat type kernel equation,

$$K_t - \alpha \boldsymbol{\nabla}^2 K + \frac{\beta}{2\alpha} F(\boldsymbol{r},t) K = 0, \tag{21}$$

with the initial condition $K(\boldsymbol{r},0,\boldsymbol{r'},0) = \delta(\boldsymbol{r}-\boldsymbol{r'})$. The solution of this equation can be expressed by the Feynman–Kac path integral formula:

$$K(\boldsymbol{r},t,\boldsymbol{r'},0) = \int [D\boldsymbol{r}] \exp\left(-\frac{S}{2\alpha}\right), \tag{22}$$

where S is the related action, i.e.,

$$S[\boldsymbol{r}(t)] = \int_0^t d\tau \left[\frac{1}{2}\dot{\boldsymbol{r}}^2 + \beta F(\boldsymbol{r},\tau)\right]. \tag{23}$$

In the case of the traveling wave solution the function $\boldsymbol{\phi}(\boldsymbol{\xi})$, where $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{u}t$, obeys the equation

$$[(\boldsymbol{\phi} - \boldsymbol{u})\boldsymbol{\nabla}]\,\boldsymbol{\phi} = \alpha \boldsymbol{\nabla}(\boldsymbol{\nabla\phi}) + \beta \boldsymbol{f}. \tag{24}$$

According to the Helmholtz theorem, the field $\boldsymbol{\phi}(\boldsymbol{\xi})$ can be split into the sum of the gradient and vortex fields

$$\boldsymbol{\phi} = \boldsymbol{\phi}_g + \boldsymbol{\phi}_v, \tag{25}$$

where $\boldsymbol{\phi}_g = \boldsymbol{\nabla}\psi$, i.e. $\boldsymbol{\nabla}\times\boldsymbol{\phi}_g = 0$, and $\boldsymbol{\phi}_v = \boldsymbol{\nabla}\times\boldsymbol{\chi}$, i.e. $\boldsymbol{\nabla}\cdot\boldsymbol{\phi}_v = 0$.

In the same way also the inhomogeneous term $\beta\boldsymbol{f}$ can be represented: $\boldsymbol{f}(\xi) = \boldsymbol{f}_g(\xi) + \boldsymbol{f}_v(\xi)$.

From equation (26) it follows that $\boldsymbol{\phi}_g(\boldsymbol{\xi})$ for $\boldsymbol{\phi}_v(\boldsymbol{\xi}) = \boldsymbol{f}_v(\boldsymbol{\xi}) = 0$ must obey the equation

$$\alpha \boldsymbol{\nabla\phi} = \frac{1}{2}\boldsymbol{\phi}^2 - (\boldsymbol{u\phi}) - \beta\varphi + \frac{1}{2}C_1, \tag{26}$$

where C_1 is the integration constant independent of $\boldsymbol{\xi}$, and $\boldsymbol{f}(\boldsymbol{\xi}) = \boldsymbol{\nabla}\varphi$.

Let $\psi = \psi(\boldsymbol{\xi})$ be the solution of the three-dimensional Schrödinger equation

$$\triangle\psi + (C_2 + a\varphi)\psi = 0. \tag{27}$$

Then the gradient part of $\phi(\xi)$

$$\boldsymbol{\phi}(\boldsymbol{\xi}) = \boldsymbol{\phi}(\boldsymbol{x} - \boldsymbol{u}t) = -2\alpha\boldsymbol{\nabla}\ln\psi + \boldsymbol{u} \tag{28}$$

is the solution of equation (26) and, consequently, of the initial VIBE (26) for

$$C_1 = \boldsymbol{u}^2 + 4\alpha^2 C_2 \qquad \text{and} \qquad \beta = -2\alpha^2 a. \tag{29}$$

Equation (26) suggests that the vortical constituent $\boldsymbol{\phi}_v(\boldsymbol{\xi})$ of the field obeys the equation

$$[(\boldsymbol{\phi} - \boldsymbol{u})\boldsymbol{\nabla}]\,\boldsymbol{\phi} = \beta\boldsymbol{f}, \tag{30}$$

where now $\boldsymbol{f} = \boldsymbol{e}^i f_i = \boldsymbol{\nabla}\times\boldsymbol{\chi}$. The solution of this equation is

$$\boldsymbol{\phi} = \boldsymbol{u} + \boldsymbol{e}^i\sqrt{2\beta F_i}, \tag{31}$$

where $\boldsymbol{e}^i$ is the unit basis vector and $\partial_i F_j = f_j\,\delta_{ij}$.

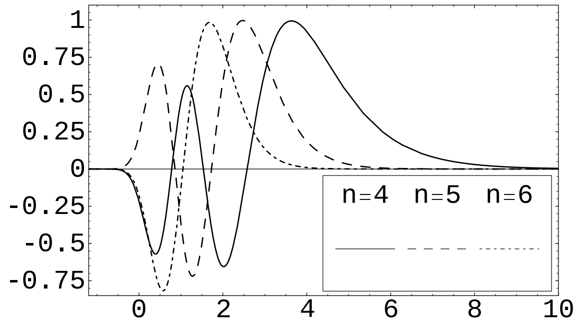

Figure 1. Solution $\phi(\xi) = -2\alpha\psi_n'/\psi_n + u$, where $\psi_n(y) = y^{(l+1)/2}e^{-\frac{1}{2}y}L_n^{l+1/2}(y)$, $y = \sqrt{\gamma/2}\xi^2$ of the VIBE (16) in the case, when $f(\xi) = \gamma\xi/2 - (l+1)/\xi$, integration constant $\alpha = \beta = 1$, $l = 7/2$ and parameter n changes from $n = 4$ to $n = 6$. All solutions are normalized to the amplitude values

Consider another potential. In the case of inhomogeneous term it looks like a three-dimensional oscillator

$$\boldsymbol{f}(\boldsymbol{\xi}) = \frac{1}{2}\gamma\boldsymbol{\xi} - \frac{l+1}{|\boldsymbol{\xi}|}\boldsymbol{e}, \qquad \boldsymbol{\xi} = \boldsymbol{r} - \boldsymbol{u}t. \tag{32}$$

Such choice of the inhomogeneous term corresponds to the potential

$$\varphi(\boldsymbol{\xi}) = \frac{1}{4}\gamma^2\boldsymbol{\xi}^2 - \frac{l(l+1)}{\boldsymbol{\xi}^2} - \gamma\left(l + \frac{3}{2}\right). \tag{33}$$

Then for the IBE

$$\boldsymbol{\phi}_t + (\boldsymbol{\phi}\boldsymbol{\nabla})\boldsymbol{\phi} - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi}) = \beta\left[\frac{1}{2}\gamma(\boldsymbol{r} - \boldsymbol{u}t) - \frac{l+1}{|\boldsymbol{r} - \boldsymbol{u}t|}\boldsymbol{e}\right] \tag{34}$$

the solution is

$$\boldsymbol{\phi}(\boldsymbol{\xi}) = -2\alpha\boldsymbol{\nabla}\psi_n/\psi_n + \boldsymbol{u}, \tag{35}$$

where $\psi_n(y) = y^{(l+1)/2}e^{-\frac{1}{2}y}L_n^{l+1/2}(y)$, $y = \sqrt{\gamma/2}\xi^2$. For a graphic representation of solution $\phi(|\boldsymbol{r} - \boldsymbol{u}t|)$, see Fig. 1.

In the case of potential (33) we have an infinite number of constants $C_2 = 2\gamma n$ and, consequently, the same infinite number of integration constants

$$C_1 = u^2 - 8\alpha^2\gamma n. \tag{36}$$

We can see that the gradient constituent $\boldsymbol{\phi}_g$ (28) of the VIBE qualitatively does not differ from the one-dimensional case (6) and has the same number of exact complete analytical solutions with the spatial variable $\boldsymbol{x} - \boldsymbol{u}t$. However, the presence of the vortical constituent $\boldsymbol{\phi}_v$ in the multi-dimensional case draws a qualitative difference between the VIBE and the one-dimensional IBE (6). Note that $\boldsymbol{\phi} = \boldsymbol{\phi}_g + \boldsymbol{\phi}_v$, because of the nonlinearity of the VIBE, is not its solution.

4 Discussion and conclusions

Exact solutions of any evolution equation are known for very limited special cases, therefore new exact solutions of KhZKE are very interesting in themselves.

Besides, the KhZKE is a limit case of a lot of mathematical models of more complicated nonlinear and dissipative systems. Exact analysis of a corresponding KhZKE provides a useful information about the behavior of such systems.

Using the known relation between the diffusion and Schrödinger equations, which is contained in diagram (11), we obtain that solution for $\boldsymbol{\nabla\varphi}(\boldsymbol{\xi})$ expresses the solution of the Schrödinger equation in the presence of nonlinearity.

$$i\boldsymbol{\phi}_t + \boldsymbol{\phi}(\boldsymbol{\nabla\phi}) - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla\phi}) = \boldsymbol{\varphi}(\boldsymbol{x} - \boldsymbol{u}t), \tag{37}$$

Sometimes, when the solutions of initial equations exhibit an exotic behavior, we can speak only about solutions of the enveloping model in the neighborhood of the solutions of initial equations. For instance, the one-dimensional equation of motion of ideal gas, as is well known, has a discontinuity in the gas flow, at the same time viscous gas has no such discontinuities, and only shock transitions at low meanings of viscosity are obtained. In this sense, the heat equation

$$\boldsymbol{\phi}_t - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla\phi}) = f(x), \tag{38}$$

describes the stationary heat distribution in a certain volume, because solutions of Poisson equation can be obtained from the heat equation in the limit of transition at $t \to \infty$. In this same sense, an IBE with the time-independent right-hand side is a covering model of a stationary nonlinear Poisson equation

$$\boldsymbol{\phi\nabla\phi} - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla\phi}) = \beta\boldsymbol{f}(\boldsymbol{x}). \tag{39}$$

This is especially actual for the sign changing coefficients α and β for so-called equations with changing parabolicity [11].

The obtained KhZKE solutions, because of their general character, allow a wide range of applications. As a concrete example, it is quite appropriate to mention the Kardar–Parisi–Zhang (KPZ) equation in (1+1)-dimension systems and crystal growth [12], the nonlinear dynamics of a moving line [13], galaxy formations [14, 15, 9], behavior of magnetic flux line in superconductor [16], and spin glasses [17]. Numerous examples of the applications are presented in [18].

Finally, exact solutions can be considered as a test model for the very promising and actively developing field of computer simulations [19].

Exact solutions of the Schrödinger equation are known to be related to the internal symmetry of a corresponding Hamiltonian [20]. As follows from the considered above subject, the algebra of supersymmetry should exist also in nonlinear and inhomogeneous cases of KhZKE, which in the physical sense is far from obvious.

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