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Invariance of Quasilinear Equations of Hyperbolic Type with Respect to Three-Dimensional Lie Algebras

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We have completely solved the problem of description of quasi-linear hyperbolic differential equations in two independent variables, that are invariant under three-parameter Lie groups.

The problem of group classification of differential equations is one of the central problems of modern symmetry analysis of differential equations [1]. One of the important classes are hyperbolic equations. The problem of group classification of such equations was discussed by many authors (see for instance [2–6]). In this paper we consider the problem of the group classification of equations of the form:

$$u_{tt} = u_{xx} + F(t, x, u, u_x), \tag{1}$$

where $u = u(t, x)$ and F is an arbitrary nonlinear differentiable function, with $F_{u_x,u_x} \neq 0$ is an arbitrary nonlinear smooth function, which dependent variables u or u_x. We use the following notation $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $F_{u_x} = \frac{\partial F}{\partial u_x}$, $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$. For the group classification of equation (1) we use the approach proposed in [7]. Here we give three main results (for details, the reader is referred to [8]).

Theorem 1. *The infinitesimal operator of the symmetry group of the equation (1) has the following form:*

$$X = (\lambda t + \lambda_1)\partial_t + (\lambda x + \lambda_2)\partial_x + (h(x)u + r(t, x))\partial_u, \tag{2}$$

where λ, λ_1, λ_2 are arbitrary real constants and $h(x)$, $r(t, x)$ are arbitrary functions which satisfy the condition

$$\begin{aligned} & r_{tt} - \frac{d^2 h}{dx^2}u - r_{xx} + (h - 2\lambda)F - (\lambda t + \lambda_1)F_t - (\lambda x + \lambda_2)F_x \\ & \quad - (hu + r)F_u - 2u_x\frac{dh}{dx} - u_x(h - \lambda)F_{u_x} - \frac{dh}{dx}uF_{u_x} - r_xF_{u_x} = 0. \end{aligned} \tag{3}$$

Theorem 2. *The equivalence group of the equation (1) is given by transformations of the following form:*

$$\bar{t} = \gamma t + \gamma_1, \qquad \bar{x} = \epsilon\gamma x + \gamma_2, \qquad v = \rho(x)u + \theta(t, x), \tag{4}$$

$\gamma \neq 0$, $\rho \neq 0$, $\epsilon = \pm 1$.

Theorem 3. *In the class of operators (2), there are no realizations of the algebras $so(3)$ and $sl(2, \mathbb{R})$.*

From this theorem we obtain the following:

Note 1. In the class of operators (2) there are no realizations of any real semi-simple Lie algebras.

Note 2. There are no equations (1) which have algebras of invariance, isomorphic by real semi-simple algebras, or contain those algebras as subalgebras.

The set of three-dimensional solvable Lie algebras consists of the following two decomposable Lie algebras:

$$A_{3.1} = A_1 \oplus A_1 \oplus A_1 = 3A_1; \qquad A_{3.2} = A_{2.2} \oplus A_1, \quad [e_1, e_2] = e_2,$$

and the following seven of non-decomposable Lie algebras:

$$\begin{aligned}
&A_{3.3}: \quad [e_2, e_3] = e_1;\\
&A_{3.4}: \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2;\\
&A_{3.5}: \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2;\\
&A_{3.6}: \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2;\\
&A_{3.7}: \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = qe_2, \quad (0 < |q| < 1);\\
&A_{3.8}: \quad [e_1,\ e_3] = -e_2, \quad [e_2,\ e_3] = e_1;\\
&A_{3.9}: \quad [e_1,\ e_3] = qe_1 - e_2, \quad [e_2,\ e_3] = e_1 + qe_2,\ (q > 0).
\end{aligned}$$

We give the realizations of the algebras $A_{3.3}$, $A_{3.4}$, $A_{3.5}$, $A_{3.9}$ and the corresponding values of the functions F in the equation (1). Here we find only equations, which are non-equivalent to equations of the form

$$u_{tt} = u_{xx} - u^{-1}u_x^2 + A(x)u_x + B(x)u \ln|u| + uD(t, x),$$

and which was classified in [8].

$$\begin{aligned}
&A^1_{3.3} = \langle u\partial_u,\ \partial_x,\ m\partial_t + xu\partial_u\rangle,\ m \neq 0: \quad F = -u^{-1}u_x^2 + u\tilde{G}(\omega), \quad \omega = t - mu_xu^{-1};\\
&A^2_{3.3} = \langle \partial_u,\ \partial_x,\ m\partial_t + x\partial_u\rangle,\ m \neq 0: \quad F = \tilde{G}(\omega), \quad \omega = mu_x - t;\\
&A^3_{3.3} = \langle \partial_u,\ \partial_t,\ \partial_x + t\partial_u\rangle: \quad F = \tilde{G}(u_x);\\
&A^4_{3.3} = \left\langle u\partial_u,\ \partial_t + k\partial_x,\ m\partial_t + \frac{1}{k}xu\partial_u\right\rangle,\ k > 0,\ m \in \mathbb{R}:\\
&\qquad F = -u^{-1}u_x^2 + u\tilde{G}(\omega), \quad \omega = x - kt + mku^{-1}u_x;\\
&A^5_{3.3} = \left\langle e^{mt}\partial_u, \partial_x, \partial_t + (mu + xe^{mt})\partial_u\right\rangle,\ (m > 0):\\
&\qquad F = m^2u + e^{mt}\tilde{G}(\omega), \quad \omega = e^{-mt}u_x - t;\\
&A^6_{3.3} = \langle \partial_u,\ \partial_t,\ t\partial_u\rangle: \quad F = \tilde{G}(x, u_x);\\
&A^7_{3.3} = \left\langle u\partial_u,\ \partial_t - \beta^{-1}xu\partial_u,\ \partial_t + \beta\partial_x\right\rangle,\ \beta > 0:\\
&\qquad F = -u^{-1}u_x^2 + u\tilde{G}(\omega), \quad \omega = x - \beta t - \beta^2u_xu^{-1};\\
&A^8_{3.3} = \langle u\partial_u,\ \partial_t - xu\partial_u,\ \partial_x\rangle: \quad F = t^2u + 2tu_x + u\tilde{G}(\omega), \quad \omega = t + u_xu^{-1};\\
&A^9_{3.3} = \left\langle e^{kt}\partial_u,\ \partial_t + ku\partial_u,\ \beta\partial_x + te^{kt}\partial_u\right\rangle,\ \beta > 0,\ k > 0:\\
&\qquad F = k^2u + \frac{2kx}{\beta}e^{kt} + e^{kt}\tilde{G}(\omega), \quad \omega = e^{-kt}u_x;\\
&A^{10}_{3.3} = \left\langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u,\ t\partial_t + x\partial_x + \frac{1}{2}u\partial_u\right\rangle:\\
&\qquad F = -\frac{u}{4}t^{-2} + u_x^3\tilde{G}(\omega, v), \quad \omega = tx^{-1},\ v = xu_x^2;\\
&A^{11}_{3.3} = \langle \partial_u, -t\partial_u,\ \partial_t + k\partial_x\rangle,\ k > 0: \quad F = \tilde{G}(\omega, u_x), \quad \omega = x - kt;
\end{aligned}$$

$$A^1_{3.4} = \left\langle \eta^{m-1}\partial_u,\ \partial_t + k\partial_x,\ t\partial_t + x\partial_x + \left(mu + t\eta^{m-1}\right)\partial_u \right\rangle,$$
$$\eta = x - kt,\quad k \geq 0,\quad m \in \mathbb{R},\quad m \neq 2:$$
$$F = (k^2-1)(m-1)(m-2)\eta^{-2}u - \frac{2k(1-m)}{2m-4}\eta^{m-2} + \eta^{2-m}\tilde{G}(\omega),$$
$$\omega = ((1-m)u + \eta u_x)\eta^{3m-4};$$
$$A^2_{3.4} = \left\langle e^{ktx^{-1}}\partial_u,\ \partial_t + kx^{-1}u\partial_u,\ t\partial_t + x\partial_x + \left(u + te^{ktx^{-1}}\right)\partial_u \right\rangle,\ k \neq 0:$$
$$F = u\left(k^2t^2x^{-4} - 2ktx^{-3} + k^2x^{-2}\right) + 2ktu_x x^{-2} + e^{ktx^{-1}}\left(2k\ln|x|x^{-1} + x^{-1}\tilde{G}(\omega)\right),$$
$$\omega = e^{-ktx^{-1}}\left(u_x + ktux^{-2}\right);$$
$$A^3_{3.4} = \left\langle kx^{-1}u\partial_u,\ \partial_t - k\ln|x|x^{-1}u\partial_u,\ t\partial_t + x\partial_x \right\rangle,\ k > 0:$$
$$F = k^2t^2ux^{-4} - 3ktux^{-3} + 2ktu_x x^{-2} + 2ktux^{-3}\ln|u|$$
$$- 2ux^{-2}\ln|u| + 2u_x x^{-1}\ln|u| + x^{-2}u\ln^2|u| + ux^{-2}\tilde{G}(\omega),$$
$$\omega = xu_x u^{-1} + \ln|u| + ktx^{-1};$$
$$A^4_{3.4} = \left\langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u,\ t\partial_t + x\partial_x + \frac{3}{2}u\partial_u \right\rangle:$$
$$F = -\frac{u}{4}t^{-2} + u_x^{-1}\tilde{G}(\omega, v),\quad \omega = tx^{-1},\quad v = x^{-1}u_x^2;$$
$$A^5_{3.4} = \langle \partial_u, -t\partial_u,\ \partial_t + k\partial_x + u\partial_u \rangle, k > 0:$$
$$F = u_x\tilde{G}(\omega, v),\quad \omega = x - kt,\quad v = \ln|u_x| - t;$$
$$A^1_{3.5} = \left\langle \eta^{m-1}\partial_u,\ \partial_t + k\partial_x,\ t\partial_t + x\partial_x + mu\partial_u \right\rangle,\ \eta = x - kt,\ k > 0,\ m \in \mathbb{R}:$$
$$F = \left(k^2-1\right)(m-1)(m-2)u\eta^{-2} + \eta^{m-2}\tilde{G}(\omega),\quad \omega = ((1-m)u + u_x\eta)\eta^{-m};$$
$$A^2_{3.5} = \left\langle \partial_x,\ |t|^{m-1}\partial_u,\ t\partial_t + x\partial_x + mu\partial_u \right\rangle,\ m \in \mathbb{R}:$$
$$F = \left(2u - 3mu - m^2u\right)t^{-2} + t^{m-2}\tilde{G}(\omega),\quad \omega = u_x t^{m-1};$$
$$A^3_{3.5} = \langle \partial_t,\ \partial_x,\ t\partial_t + x\partial_x \rangle:\quad F = u_x^2\tilde{G}(u);$$
$$A^4_{3.5} = \langle \partial_t,\ \partial_x,\ t\partial_t + x\partial_x + mu\partial_u \rangle,\ m \neq 0,1,2:\quad F = |u|^{\frac{m-2}{m}}\tilde{G}(\omega),\quad \omega = u_x^{-1}|u|^{\frac{m-1}{m}};$$
$$A^5_{3.5} = \langle \partial_t,\ \partial_x,\ t\partial_t + x\partial_x + \partial_u \rangle,\ m \neq 0:\quad F = e^{-2u}\tilde{G}(\omega),\quad \omega = e^u u_x;$$
$$A^6_{3.5} = \left\langle \partial_t,\ x^{-1}u\partial_u,\ t\partial_t + x\partial_x \right\rangle,\ k \neq 0:$$
$$F = 2u_x x^{-1}\ln|u| + u\ln^2|u|x^{-2} - 2u\ln|u|x^{-2} + x^{-2}u\tilde{G}(\omega),\quad \omega = u_x u^{-1}x + \ln|u|;$$
$$A^7_{3.5} = \left\langle \partial_t + kx^{-1}u\partial_u,\ e^{ktx^{-1}}\partial_u,\ t\partial_t + x\partial_x + u\partial_u \right\rangle,\ k \in \mathbb{R}:$$
$$F = uk\left(kt^2 - 2xt + kx^2\right)x^{-4} + 2ktu_x x^{-2} + e^{ktx^{-1}}x^{-1}\tilde{G}(\omega),$$
$$\omega = e^{-ktx^{-1}}\left(u_x + ktux^{-2}\right);$$
$$A^1_{3.9} = \langle \sin(t)\partial_u,\ \cos(t)\partial_u, \partial_t + k\partial_x + qu\partial_u \rangle,\ k \geq 0,\ q > 0:$$
$$F = -u + u_x\tilde{G}(\eta, v),\quad \eta = x - kt,\ v = e^{-qt}u_x;$$
$$A^2_{3.9} = \left\langle\ |t|^{\frac{1}{2}}\sin\left(\frac{\ln|t|}{2(k-q)}\right)\partial_u, |t|^{\frac{1}{2}}\cos\left(\frac{\ln|t|}{2(k-q)}\right)\partial_u,\ 2(k-q)(t\partial_t + x\partial_x) + ku\partial_u \right\rangle,$$
$$k \in \mathbb{R},\ q > 0,\ k \neq q:\quad F = -\frac{(k-q)^2+1}{4(k-q)^2}t^{-2}u + |t|^{\frac{4q-3k}{2(k-q)}}\tilde{G}(\omega, v),$$
$$\omega = tx^{-1},\quad v = |t|^{k-2q}|u_x|^{2(k-q)}.$$

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