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Subgroups of Extended Poincaré Group and New Exact Solutions of Maxwell Equations

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Using three-parameter subgroups of the extended Poincaré group $\tilde{P}(1,3)$ we have constructed ansatzes reducing the Maxwell equations to systems of ordinary differential equations. This enables us to construct a number of new exact solutions of the Maxwell equations.

1 Introduction

The electromagnetic field is described by the electric $\boldsymbol{E} = \boldsymbol{E}(x_0, \boldsymbol{x})$ and magnetic $\boldsymbol{H} = \boldsymbol{H}(x_0, \boldsymbol{x})$ fields. In the absence of charges, we have the system of vacuum Maxwell equations

$$\operatorname{rot} \boldsymbol{E} = -\frac{\partial \boldsymbol{H}}{\partial x_0}, \qquad \operatorname{div} \boldsymbol{H} = 0, \qquad \operatorname{rot} \boldsymbol{H} = \frac{\partial \boldsymbol{E}}{\partial x_0}, \qquad \operatorname{div} \boldsymbol{E} = 0. \tag{1}$$

As it is well-known [1, 2], the maximal point symmetry group admitted by the Maxwell equations (1) is the 16-parameter group which is the direct product of the 15-parameter conformal group $C(1,3)$ and of the one-parameter Heaviside–Larmor–Rainich group H. It contains as a subgroup the extended Poincaré group $\tilde{P}(1,3)$ generated by the following vector fields:

$$\begin{aligned}
&P_\mu = \partial_{x_\mu}, \qquad J_{0a} = x_0 \partial_{x_a} + x_a \partial_{x_0} + \varepsilon_{abc}(E_b \partial_{H_c} - H_b \partial_{E_c}),\\
&J_{ab} = x_b \partial_{x_a} - x_a \partial_{x_b} + E_b \partial_{E_a} - E_a \partial_{E_b} + H_b \partial_{H_a} - H_a \partial_{H_b},\\
&D = x_\mu \partial_{x_\mu} - 2(E_a \partial_{E_a} + H_a \partial_{H_a}).
\end{aligned} \tag{2}$$

Here $\mu = 0,1,2,3$; $a,b,c = 1,2,3$; summation over repeated indices is understood, the index μ taking the values $0,1,2,3$ and the indices a,b taking the values $1,2,3$; ε_{abc} is the totally antisymmetric third-order tensor, $\partial_{x_\mu} = \frac{\partial}{\partial x_\mu}$, $\partial_{E_a} = \frac{\partial}{\partial E_a}$, $\partial_{H_a} = \frac{\partial}{\partial H_a}$.

The large symmetry group admitted by the Maxwell equations allows one to construct many exact solutions by the symmetry reduction method [3, 4, 5, 6, 7, 8]. Using three-parameter subgroups of the Poincaré group $P(1,3)$ with generators P_μ, $J_{\mu\nu}$ (2) enabled us to obtain in [9, 10] a number of exact solutions of the system (1).

The aim of the present report is to give an exhaustive description of $\tilde{P}(1,3)$-invariant ansatzes for the Maxwell field $(\boldsymbol{E}, \boldsymbol{H})$ reducing equations (1) to systems of ordinary differential equations. Using them we will construct new exact solutions of the Maxwell equations.

Let $\tilde{p}(1,3)$ be the Lie algebra of the Poincaré group with the generators (2) and $\tilde{p}^{(1)}(1,3)$ be the Lie algebra having as basis elements

$$P^{(1)}_\mu = \partial_{x_\mu}, \qquad J^{(1)}_{\mu\nu} = x^\mu \partial_{x_\nu} - x^\nu \partial_{x_\mu}, \qquad D^{(1)}_\mu = x_\mu \partial_{x_\mu},$$

where $\mu, \nu = 0,1,2,3$; lowering of the indices μ, ν is performed with the help of the metric tensor of the Minkowski space-time $g_{\mu\nu}$.

Next, let L be a subalgebra of the algebra $\tilde{p}(1,3)$ having rank r, and let the projection of the algebra L onto $\tilde{p}^{(1)}(1,3)$ have rank $r^{(1)}$. It follows from the general theory of invariant solutions of differential equations ([3]) that subalgebras of the algebra L satisfying the additional condition $r = r^{(1)} = 3$ give rise to ansatzes reducing (1) to systems of ordinary differential equations. It is not difficult to see that in the case $\dim L = 3$ and a basis of functionally independent invariants of the algebra L consists of seven functions $\Omega_i = \Omega_i(x_0, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H})$ $(i = 1, 2, \ldots, 6)$ and $\omega = \omega(x_0, \boldsymbol{x})$. The structure of an invariant ansatz is completely determined by the form of the functions Ω_i.

Let us introduce the notations

$$\boldsymbol{V} = (E_1\ E_2\ E_3\ H_1\ H_2\ H_3)^T, \qquad \boldsymbol{W} = \left(\tilde{E}_1\ \tilde{E}_2\ \tilde{E}_3\ \tilde{H}_1\ \tilde{H}_2\ \tilde{H}_3\right)^T.$$

Then the general form of the basis elements of the three-dimensoinal Lie algebra $L = \langle X_a | a = 1, 2, 3\rangle$ reads as

$$X_a = \xi_{a\mu}(x_0, \boldsymbol{x})\partial_{x_\mu} + \rho_{alk} V_k \partial_{V_l}.$$

Here, and in the following, $m, n, k, l = 1, 2, \ldots, 6$; $\mu, \nu = 0, 1, 2, 3$.

As the basis elements (2) realize a linear representation of the algebra $\tilde{p}(1, 3)$ and, the condition $r = r^{(1)}$ holds, the general form of an ansatz invariant with respect to a three-dimensional subalgebra $L \in \tilde{p}(1, 3)$ reads [8, 9, 10]

$$\boldsymbol{V} = \Lambda \boldsymbol{W}(\omega), \tag{3}$$

where $\Lambda = \Lambda(x_0, \boldsymbol{x})$ is a 6×6 matrix nonsingular in some domain of the space $\mathbb{R}_{0,3} = \{(x_0, \boldsymbol{x}) : x_\mu \in \mathbb{R},\ \mu = 0, 1, 2, 3\}$ which, together with a smooth scalar function $\omega = \omega(x)$, satisfies the following system of partial differential equations:

$$\xi_{a\mu} \frac{\partial \Lambda_{mn}}{\partial_{x_\mu}} + f_{ml} \rho_{aln} = 0, \tag{4}$$

$$\xi_{a\mu} \frac{\partial \omega_{mn}}{\partial_{x_\mu}} = 0. \tag{5}$$

Here the symbol Λ_{mn} stands for the (m, n) entry of the matrix Λ.

Thus, the problem of symmetry reduction of the Maxwell equations by scale-invariant ansatzes contains as a subproblem integration of systems of the form (4), (5) for each inequivalent three-dimensional algebra. Remarkably, there is no need to consider all inequivalent algebras, since the following results hold:

Lemma 1 ([9]). *Let* $\boldsymbol{E}$, $\boldsymbol{H}$ *be functions of* x_1, x_2, $\xi = \frac{1}{2}(x_0 - x_3)$ *only. Then the Maxwell equations can be integrated, and their general solution is given by*

$$E_1 = \frac{1}{2}(R + R^* + T_1 + T_1^*), \qquad E_2 = \frac{1}{2}(iR - iR^* + T_2 + T_2^*), \qquad E_3 = S + S^*,$$
$$H_1 = \frac{1}{2}(iR - iR^* - T_2 - T_2^*), \qquad E_2 = \frac{1}{2}(R + R^* - T_1 - T_1^*), \qquad E_3 = iS - iS^*,$$

where $T_a = \frac{\partial^2 \sigma_a}{\partial \xi^2}$, $a = 1, 2$; $S = \frac{\partial \sigma_1}{\partial \xi} + i\frac{\partial \sigma_2}{\partial \xi} + \lambda(z)$, $R = -2\left(\frac{\partial \sigma_1}{\partial z} + i\frac{\partial \sigma_2}{\partial z}\right) + \frac{d\lambda}{dz}\xi$; $\sigma = \sigma_a(z, \xi)$, $z = x_1 + ix_2$ *and* $\lambda = \lambda(z)$ *are arbitrary analytic functions.*

Lemma 2 ([11]). *Let* $\boldsymbol{E}$, $\boldsymbol{H}$ *be functions of* x_0, x_3 *only. Then the Maxwell equations can be integrated, and their general solution is given by the formulae below*

$$E_1 = f_1(\xi) + g_1(\eta), \qquad E_2 = f_2(\xi) + g_2(\eta), \qquad E_3 = C_1,$$
$$H_1 = f_2(\xi) - g_2(\eta), \qquad H_2 = -f_1(\xi) + g_1(\eta), \qquad H_3 = C_2,$$

where f_1, f_2, g_1, g_2 *are arbitrary smooth functions,* $\xi = x_0 - x_3$, $\eta = x_0 + x_3$ *and* C_1, C_2 *are arbitrary real constants.*

Consequently, to obtain new solutions of the Maxwell equations it is sufficient to restrict our considerations to those three-dimensional subalgebras of $\tilde{p}(1,3)$ which are not conjugate to subalgebras of $p(1,3)$ and, in addition, fulfill the conditions

$$1)\ \ r = r^{(1)} = 3; \qquad 2)\ \ \langle P_0 \pm P_3\rangle \not\subset L, \quad \langle P_0, P_3\rangle \not\subset L; \qquad 3)\ \ \langle P_1, P_2\rangle \not\subset L.$$

Making use of the classification of inequivalent subalgebras of the algebra $\tilde{p}(1,3)$ obtained in [9, 10] we have checked that the above conditions are satisfied by the following seven subalgebras [11]:

$$\begin{aligned}
&L_1 = \langle J_{12}, D, P_0\rangle; \qquad L_2 = \langle J_{12}, D, P_3\rangle; \qquad L_3 = \langle J_{03}, D, P_1\rangle;\\
&L_4 = \langle J_{03}, J_{12}, D\rangle; \qquad L_5 = \langle G_1, J_{03} + \alpha D, P_2\rangle \ \ (0 < |\alpha| \le 1);\\
&L_6 = \langle J_{03} - D + P_0 + P_3, G_1, P_2\rangle; \qquad L_7 = \langle J_{03} + 2D, G_1 + P_0 - P_3, P_2\rangle,
\end{aligned}$$

where $G_1 = J_{01} - J_{13}$.

As direct verification shows, the basis elements of the above algebras satisfy the condition $r = r^{(1)} = 3$. Consequently, each of them gives rise to an ansatz of the type given in (3). Furthermore, these ansatzes can be represented in a unified way, namely

$$\begin{aligned}
E_1 &= \theta\{(\tilde{E}_1\cos\theta_3 - \tilde{E}_2\sin\theta_3)\cosh\theta_0 + (\tilde{H}_1\sin\theta_3 + \tilde{H}_2\cos\theta_3)\sinh\theta_0\\
&\quad + 2\theta_1\tilde{E}_3 + 2\theta_2\tilde{H}_3 + 4\theta_1\theta_2\Sigma_1 + 2(\theta_1^2 - \theta_2^2)\Sigma_2\},\\
E_2 &= \theta\{(\tilde{E}_2\cos\theta_3 + \tilde{E}_1\sin\theta_3)\cosh\theta_0 + (\tilde{H}_2\sin\theta_3 - \tilde{H}_1\cos\theta_3)\sinh\theta_0\\
&\quad - 2\theta_1\tilde{H}_3 + 2\theta_2\tilde{E}_3 + 4\theta_1\theta_2\Sigma_2 - 2(\theta_1^2 - \theta_2^2)\Sigma_1\},\\
E_3 &= \theta\{\tilde{E}_3 + 2\theta_1\Sigma_2 + 2\theta_2\Sigma_1\},\\
H_1 &= \theta\{(\tilde{H}_1\cos\theta_3 - \tilde{H}_2\sin\theta_3)\cosh\theta_0 - (\tilde{E}_1\sin\theta_3 + \tilde{E}_2\cos\theta_3)\sinh\theta_0\\
&\quad + 2\theta_1\tilde{H}_3 - 2\theta_2\tilde{E}_3 - 4\theta_1\theta_2\Sigma_2 + 2(\theta_1^2 - \theta_2^2)\Sigma_1\},\\
H_2 &= \theta\{(\tilde{H}_2\cos\theta_3 + \tilde{H}_1\sin\theta_3)\cosh\theta_0 + (\tilde{E}_1\cos\theta_3 - \tilde{E}_2\sin\theta_3)\sinh\theta_0\\
&\quad + 2\theta_1\tilde{E}_3 + 2\theta_2\tilde{H}_3 + 4\theta_1\theta_2\Sigma_1 + 2(\theta_1^2 - \theta_2^2)\Sigma_2\},\\
H_3 &= \theta\{\tilde{H}_3 + 2\theta_1\Sigma_1 - 2\theta_2\Sigma_2\},
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_1 &= [(\tilde{H}_2 - \tilde{E}_1)\sin\theta_3 - (\tilde{E}_2 + \tilde{H}_1)\cos\theta_3]e^{-\theta_0},\\
\Sigma_1 &= [(\tilde{E}_2 + \tilde{H}_1)\sin\theta_3 + (\tilde{H}_2 - \tilde{E}_1)\cos\theta_3]e^{-\theta_0},
\end{aligned}$$

and the functions $\theta = \theta(x_0, \boldsymbol{x})$, $\theta_\beta = \theta_\beta(x_0, \boldsymbol{x})$ $(\beta = 0, 1, 2)$, $\omega = \omega(x_0, \boldsymbol{x})$ are ([11]):

$$\begin{aligned}
&L_1 : \theta = x_3^2, \ \ \theta_1 = \arctan\frac{x_2}{x_1}, \ \ \theta_0 = \theta_2 = 0, \ \ \omega = \frac{x_1^2 + x_2^2}{x_3^2};\\
&L_2 : \theta = x_0^2, \ \ \theta_1 = \arctan\frac{x_2}{x_1}, \ \ \theta_0 = \theta_2 = 0, \ \ \omega = \frac{x_1^2 + x_2^2}{x_0^2};\\
&L_3 : \theta = x_2^2, \ \theta_0 = \ln\left|(x_0 + x_3)x_2^{-1}\right|, \ \ \theta_1 = \theta_2 = 0, \ \ \omega = \left(x_0^2 - x_3^2\right)x_2^{-2};\\
&L_4 : \theta = x_0^2 - x_3^2, \ \ \theta_0 = \frac{1}{2}\ln\left|(x_0 + x_3)(x_0 - x_3)^{-1}\right|, \ \ \theta_1 = \arctan\frac{x_2}{x_1}, \ \ \theta_2 = 0,\\
&\qquad \omega = \left(x_1^2 + x_2^2\right)\left(x_0^2 - x_3^2\right)^{-1};
\end{aligned}$$

$$L_5: 1)\ \ \theta = x_0 - x_3, \quad \theta_0 = -\frac{1}{2}\ln|x_0 - x_3|, \quad \theta_1 = 0, \quad \theta_2 = -\frac{1}{2}x_1 (x_0 - x_3)^{-1},$$
$$\omega = x_0 + x_3 - x_1^2(x_0 - x_3)^{-1} \quad \text{for} \quad \alpha = -1;$$
$$2)\ \ \theta = x_0^2 - x_1^2 - x_3^2,\ \theta_0 = \frac{1}{2\alpha}\ln\left|x_0^2 - x_1^2 - x_3^2\right|, \quad \theta_1 = 0, \quad \theta_2 = -\frac{1}{2}x_1(x_0 - x_3)^{-1},$$
$$\omega = 2\alpha\ln|x_0 - x_3| + (1-\alpha)\ln\left|x_0^2 - x_1^2 - x_3^2\right| \quad \text{for} \quad \alpha \neq -1;$$
$$L_6: \theta = x_0 - x_3, \quad \theta_0 = -\frac{1}{2}\ln|x_0 - x_3|, \quad \theta_1 = 0, \quad \theta_2 = -\frac{x_1}{2(x_0 - x_3)},$$
$$\omega = x_0 + x_3 - x_1^2(x_0 - x_3)^{-1} + \ln|x_0 - x_3|;$$
$$L_7: \theta = \left(4x_1 - (x_0 - x_3)^2\right)^2, \quad \theta_0 = \frac{1}{2}\ln\left|4x_1 - (x_0 - x_3)^2\right|, \quad \theta_1 = 0,$$
$$\theta_2 = -\frac{1}{4}(x_0 - x_3), \quad \omega = \left[x_0 + x_3 - x_1(x_0 - x_3) + \frac{1}{6}(x_0 - x_3)^3\right]\left|4x_1 - (x_0 - x_3)^2\right|^{-\frac{3}{2}}.$$

Substituting the ansatzes obtained in this way into the initial system (1) yields systems of ordinary differential equations for the unknown functions $\tilde{E}_a$, $\tilde{H}_a$ $(a = 1, 2, 3)$. If, for example, we take the ansatz invariant under the algebra L_1 and insert it into the Maxwell equations, then, after some algebraic manipulations, we obtain the following system for $\tilde{E}_a(\omega)$, $\tilde{H}_a(\omega)$ $(a = 1, 2, 3)$:

$$2\omega(1+\omega)\ddot{\tilde{E}}_3 + (7\omega + 2)\dot{\tilde{E}}_3 + 3\tilde{E}_3 = 0, \qquad 2\omega(1+\omega)\ddot{\tilde{H}}_3 + (7\omega + 2)\dot{\tilde{H}}_3 + 3\tilde{H}_3 = 0,$$
$$f = h = -2\sqrt{\omega}(\tilde{E}_3 + (1+\omega)\dot{\tilde{E}}_3), \qquad g = -\rho = 2\sqrt{\omega}(\tilde{H}_3 + (1+\omega)\dot{\tilde{H}}_3),$$

where

$$f = \tilde{E}_1 + \tilde{H}_2, \qquad g = \tilde{E}_2 - \tilde{H}_1, \qquad h = \tilde{E}_1 - \tilde{H}_2,$$
$$\rho = \tilde{E}_2 + \tilde{H}_1, \qquad \dot{\tilde{E}}_3 = \frac{d\tilde{E}_3}{d\omega}, \qquad \ddot{\tilde{E}}_3 = \frac{d^2\tilde{E}_2}{d\omega^2}.$$

Taking into account that we have $\omega \geq 0$, we represent the general solution of the above system as follows

$$\tilde{E}_3 = (1+\omega)^{-\frac{3}{2}}\left[C_1\left(\ln\left|\frac{\sqrt{1+\omega}-1}{\sqrt{1+\omega}+1}\right| + 2\sqrt{1+\omega}\right) + C_2\right],$$
$$\tilde{H}_3 = (1+\omega)^{-\frac{3}{2}}\left[C_3\left(\ln\left|\frac{\sqrt{1+\omega}-1}{\sqrt{1+\omega}+1}\right| + 2\sqrt{1+\omega}\right) + C_4\right],$$

where C_1, C_2, C_3, C_4 are integration constants, and we easily get the corresponding exact solutions of the Maxwell equations (1):

$$E_a = -\frac{2C_1x_a}{x_3\left(x_1^2 + x_2^2\right)} + x_a\sigma^{-\frac{3}{2}}A_{12}, \qquad E_3 = x_3\sigma^{-\frac{3}{2}}A_{12},$$
$$H_a = -\frac{2C_3x_a}{x_3\left(x_1^2 + x_2^2\right)} + x_a\sigma^{-\frac{3}{2}}A_{34}, \qquad H_3 = x_3\sigma^{-\frac{3}{2}}A_{34}.$$

Here $A_{ij} = C_i\left(\ln\left|\frac{\sqrt{\sigma}-x_3}{\sqrt{\sigma}+x_3}\right| + 2x_3^{-1}\sqrt{\sigma}\right) + C_j$, $\sigma = x_1^2 + x_x^2 + x_3^2$, $a = 1, 2$.

Let us note that the systems of ordinary differential equations obtained via reduction of the Maxwell equations by ansatzes invariant under the remaining algebras L_2–L_7 are also integrable in terms of elementary functions.

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