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On Four Orthogonal Projections that Satisfy the Linear Relation $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = I$, $\alpha_i > 0$

Stanislav KRUGLYAK † and *Anatolii KYRYCHENKO* ‡

† *National Academy of Security Service of Ukraine, Kyiv, Ukraine*

‡ *Kyiv National University of Building and Architecture, 31 Povitroflotsky Prosp., Kyiv 03037, Ukraine*
E-mail: *AAKirichenko@rambler.ru*

In the article we investigate the sets of orthogonal projections which satisfy the linear relation $\sum\limits_{i=1}^{n} \alpha_i P_i = I$, $\alpha_i > 0$, up to unitary equivalence. A problem of unitary classification of four projections that satisfy the linear relation $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = I$, $\alpha_i > 0$ is considered in [1–4]. We present a new method for solving this problem that is based on functors of Coxeter, which are analogous to those introduced in [5].

Let $\mathfrak{P}_{n,\vec{\alpha}} = \mathbb{C}\langle p_1, p_2, \ldots, p_n \,|\, p_i^2 = p_i = p_i^*, \sum\limits_{i=1}^{n} \alpha_i p_i = e\rangle$ be a $*$-algebra, where the vector $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i > 0$, $i = 1, \ldots, n$; $A = \sum\limits_{i=1}^{n} \alpha_i$. We study its representations, up to unitary equivalence, in the category of Hilbert spaces. Define Σ_n as a set of $\vec{\alpha}$ such that the category of representations $\operatorname{Rep}\mathfrak{P}_{n,\vec{\alpha}}$ is not empty.

1. Let us consider some properties of $\mathfrak{P}_{n,\vec{\alpha}}$.

Lemma 1. *If $\vec{\alpha} \in \Sigma_n$ then $A \geq 1$.*

Proof. Let π be a representation of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$: $\sum\limits_{i=1}^{n} \alpha_i \pi(p_i) = I$ then $\sum\limits_{i=1}^{n} \alpha_i (I - \pi(p_i)) = (A-1)I$. Since the operator at the left hand-side is positive then $A \geq 1$. ∎

Lemma 2. *If $A = 1$ then $\vec{\alpha} \in \Sigma_n$ and the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ has (up to unitary equivalence) only one irreducible representation $\pi : \pi(p_i) = 1$.*

Proof. If $A = 1$ then $\sum\limits_{i=1}^{n} \alpha_i (I - \pi(p_i)) = 0$ and for all $i = 1, \ldots, n$: $\pi(p_i) = I$. ∎

Definition 1. The algebra $\mathfrak{P}_{n,\vec{\alpha}}$ and the vector $\vec{\alpha}$ are called reduced if there exists such a number i_0 that for all representations π of the algebra we have $\pi(p_{i_0}) = 0$ or there exists a number j_0 that for all representations π of the algebra we have $\pi(p_{j_0}) = I$.

Remark 1. In the case of mapping of a reduced algebra to its enveloping C^*-algebra the elements p_{i_0} and $p_{j_0} - e$ belong to the $*$-radical, and the corresponding C^*-algebra will be generated by less than n linear connected projections.

Lemma 3. *If $\vec{\alpha} \in \Sigma_n : \exists\, \alpha_{i_0} > 1$ then for all representations π of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$: $\pi(p_{i_0}) = 0$, e.g. the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ is reduced.*

Proof. Take an arbitrary representation π of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ then $\sum\limits_{i \neq i_0} \alpha_i \pi(p_i) = I - \alpha_{i_0}\pi(p_{i_0})$. The operator at the left-hand side is positive. But the operator at the right-hand side is positive when $\pi(p_{i_0}) = 0$ only. ∎

Lemma 4. *If $\vec{\alpha} \in \Sigma_n$ and the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ is not reduced then $A \leq n$.*

Proof. If $A > n$, then there exists a number $i_0 : \alpha_{i_0} > 1$ and according to the Lemma 3 the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ will be reduced. ■

Let $\Sigma_n^1 = \Sigma_n \bigcap (0,1)^n$ e.g. Σ_n^1 consists of such points $\vec{\alpha} \in \Sigma_n$ that $0 < \alpha_i < 1$.

Our aim is to describe the set Σ_n^1 $(1 \leq A < n)$ and the set of representations of corresponding algebras. There are reduced and nonreduced ones among such class of algebras.

We define functors S and T (analogy with [5]), which act on the set of categories $\operatorname{Rep} \mathfrak{P}_{n,\vec{\alpha}}$. They are equivalences of categories (if $\operatorname{Rep} \mathfrak{P}_{n,\vec{\alpha}}$ is not empty, then $S(\operatorname{Rep} \mathfrak{P}_{n,\vec{\alpha}})$ (or $T(\operatorname{Rep} \mathfrak{P}_{n,\vec{\alpha}})$) is not empty and they are equivalent).

Let us define the functor T (*functor of hyperbolic reflection*).

Let $\alpha \in \Sigma_n$, $A > 1$, $\pi \in \operatorname{Rep} \mathfrak{P}_{n,\vec{\alpha}}$, then $\sum\limits_{i=1}^{n} \alpha_i \pi(p_i) = I$ and $\sum\limits_{i=1}^{n} \alpha_i (I - \pi(p_i)) = (A-1)I$ or $\sum\limits_{i=1}^{n} \frac{\alpha_i}{A-1}(I - \pi(p_i)) = I$. Define $T(\pi)(p_i) = I - \pi(p_i)$. Thus, we obtain the functor

$$T : \operatorname{Rep} \mathfrak{P}_{n,(\alpha_1,\alpha_2,\ldots,\alpha_n)} \to \operatorname{Rep} \mathfrak{P}_{n,\left(\frac{\alpha_1}{A-1},\frac{\alpha_2}{A-1},\ldots,\frac{\alpha_n}{A-1}\right)}$$

which is defined when $A > 1$.

It is easy to check that this functor is equivalence of categories (the corresponding algebras are isomorphic).

Let us define the functor S (*functor of linear reflection*).

Let $\vec{\alpha} \in \Sigma_n^1$, $\sum\limits_{i=1}^{n} \alpha_i \pi(p_i) = I$ and π be a representation of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ in the Hilbert space H_0. Since $\pi(p_i)$ is a projection then $\pi(p_i) = \Gamma_i \Gamma_i^*$, where Γ_i is the natural isometry of the space $H_i = \operatorname{Im} \pi(p_i)$ to H_0.

Let $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$. Define the linear operator $\Gamma : H \to H_0$ that is given by the matrix

$$\Gamma = \begin{pmatrix} \sqrt{\alpha_1}\,\Gamma_1 & \sqrt{\alpha_2}\,\Gamma_2 & \cdots & \sqrt{\alpha_n}\,\Gamma_n \end{pmatrix}.$$

Since $\Gamma\Gamma^* = \sum\limits_{i=1}^{n} \alpha_i \Gamma_i \Gamma_i^* = \sum\limits_{i=1}^{n} \alpha_i \pi(p_i) = I_{H_0}$, Γ^* is a partial isometry from H_0 to H. Let $\hat{H}_0 = (\operatorname{Im} \Gamma^*)^{\perp}$ and Δ^* is the natural isometry of $\hat{H}_0$ to H then $U^* = (\Gamma^*, \Delta^*)$ be a unitary operator from $\hat{H}_0 \oplus H_0$ to H. As $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$, the operators Δ and U have the Peirce decomposition

$$\Delta = \begin{pmatrix} \sqrt{1-\alpha_1}\Delta_1 & \sqrt{1-\alpha_2}\Delta_2 & \cdots & \sqrt{1-\alpha_n}\Delta_n \end{pmatrix},$$
$$U = \begin{pmatrix} \sqrt{\alpha_1}\,\Gamma_1 & \sqrt{\alpha_2}\,\Gamma_2 & \cdots & \sqrt{\alpha_n}\,\Gamma_n \\ \sqrt{1-\alpha_1}\Delta_1 & \sqrt{1-\alpha_2}\Delta_2 & \cdots & \sqrt{1-\alpha_n}\Delta_n \end{pmatrix}.$$

Since U is a unitary operator and $\Gamma_i^* \Gamma_i = I_{H_i}$, it is easy to obtain that $\Delta_i^* \Delta_i = I_{H_i}$ and $\Delta_i \Delta_i^* = Q_i$ are orthoprojections in the space $\hat{H}_0$. From $\Delta\Delta^* = I_{\hat{H}_0}$ (Δ is an isometry) it follows that $\sum\limits_{i=1}^{n} (1-\alpha_i)\Delta_i \Delta_i^* = I_{\hat{H}_0}$, $\sum\limits_{i=1}^{n} (1-\alpha_i) Q_i = I_{\hat{H}_0}$.

Define $S : \pi \to \hat{\pi}$, where $\hat{\pi}(p_i) = Q_i$. From the condition $\sum\limits_{i=1}^{n} (1-\alpha_i) Q_i = I$ we have $\hat{\pi} \in \operatorname{Ob} \operatorname{Rep} \mathfrak{P}_{n,(1-\alpha_1,1-\alpha_2,\ldots,1-\alpha_n)}$. One can see (in analogy with [5]), that the functor

$$S : \operatorname{Rep} \mathfrak{P}_{n,(\alpha_1,\alpha_2,\ldots,\alpha_n)} \to \operatorname{Rep} \mathfrak{P}_{n,(1-\alpha_1,1-\alpha_2,\ldots,1-\alpha_n)},$$

where $0 < \alpha_i < 1$ (therefore, $0 < A < n$), is an equivalence of categories.

Let π be a representation of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ in a finite-dimensional space H. We shall call the vector $(d; d_1, d_2, \ldots, d_n)$, where $d = \dim H$, $d_i = \dim \operatorname{Im} \pi(p_i)$, the generalized dimension of the representation π.

The functors T and S induce actions on the set of vectors $\vec{\alpha}$, on sums of their coordinates A and on generalized dimensions of representations of algebras $\mathfrak{P}_{n,\vec{\alpha}}$.

It it easy to check that

$$T(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{\alpha_1}{A-1}, \frac{\alpha_2}{A-1}, \ldots, \frac{\alpha_n}{A-1}\right), \qquad T(A) = \frac{A}{A-1},$$
$$T(d; d_1, d_2, \ldots, d_n) = (d; d - d_1, d - d_2, \ldots, d - d_n),$$
$$S(\alpha_1, \alpha_2, \ldots, \alpha_n) = (1 - \alpha_1, 1 - \alpha_2, \ldots, 1 - \alpha_n), \qquad S(A) = n - A,$$
$$S(d; d_1, d_2, \ldots, d_n) = \left(\sum_{i=1}^{n} d_i - d; d_1, d_2, \ldots, d_n\right).$$

Define the functors of Coxeter as $\Phi^+ = TS$ and $\Phi^- = ST$. Φ^+ is defined when $A < n - 1$, $\vec{\alpha} \in \Sigma_n^1$. Φ^- is defined when $A > 1$, $T(\vec{\alpha}) \in (0, 1)^n$. Since $T^2 = Id$, $S^2 = Id$, then $\Phi^+\Phi^- = Id$ and $\Phi^-\Phi^+ = Id$.

Let $\Phi^{+(k)} = \Phi^+\Phi^{+(k-1)}$.

Lemma 5. $\lim\limits_{k\to\infty} \Phi^{+(k)}\left(1 + \frac{1}{n-2}\right) = \frac{n-\sqrt{n^2-4n}}{2}$ *and intervals*

$$\left[1, 1 + \tfrac{1}{n-2}\right), \ \left[1 + \tfrac{1}{n-2}, \Phi^+\left(1 + \tfrac{1}{n-2}\right)\right), \ \ldots, \ \left[\Phi^{+(k-1)}\left(1 + \tfrac{1}{n-2}\right), \Phi^{+(k)}\left(1 + \tfrac{1}{n-2}\right)\right), \ \ldots$$

do not intersect and cover the interval $\left[1, \frac{n-\sqrt{n^2-4n}}{2}\right)$.

Proof. It is easy to show that $\Phi^+(1) = 1 + \frac{1}{n-2}$ and the sequence $\Phi^{+(k)}\left(1 + \frac{1}{n-2}\right)$ is increasing. Since it is bounded by 2, the limit a of the sequence exists and it is a fixed point of the map $\Phi^+(A) = 1 + \frac{1}{n-A-1}$. From the equation $1 + \frac{1}{n-a-1} = a$ (taking into account that $a < 2$) we obtain $a = \frac{n-\sqrt{n^2-4n}}{2}$. ∎

Lemma 6. $\vec{\alpha} \in \Sigma_n^1, 0 < A \leq \frac{n}{2}$, *if and only if* $T(\vec{\alpha}) \in \Sigma_n^1$ *and* $\frac{n}{2} \leq T(A) < n$.

Proof. Obviously, the map S sets one-to-one correspondence between points of Σ_n^1 with the sum $A < n$ and points Σ_n^1 with the sum $n - A$. ∎

Lemma 7. *If* $n - 1 < A < n$ *then* $\vec{\alpha} \notin \Sigma_n^1$.

Proof. If $n - 1 < A < n$ then $0 < S(A) < 1$, whence, by the Lemma 1, $S(\vec{\alpha}) \notin \Sigma_n$ and it means that $\vec{\alpha} \notin \Sigma_n^1$. ∎

Lemma 8. *If* $\vec{\alpha} \in \Sigma_n$, $A \neq 1$ *and* $\mathfrak{P}_{n,\vec{\alpha}}$ *is not reduced then* $\frac{\alpha_i}{A-1} \leq 1$ *and* $A \geq \frac{n}{n-1}$.

Proof. If there exists a number i_0 that $\frac{\alpha_{i_0}}{A-1} > 1$, then the algebra $\mathfrak{P}_{n,T(\vec{\alpha})}$ will be reduced. Take any representation π of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$. Denote $\hat{\pi}$ as the correspondent representation of the algebra $\mathfrak{P}_{n,T(\vec{\alpha})}$ then by the lemma 3 $\hat{\pi}(p_{i_0}) = 0$, so $\pi(p_{i_0}) = I$ and $\mathfrak{P}_{n,\vec{\alpha}}$ is reduced.

If for all $i : \frac{\alpha_i}{A-1} \leq 1$ then $\frac{A}{A-1} \leq n$ and from here $A \geq \frac{n}{n-1}$. ∎

2. Now we describe Σ_n^1, when $n = 3$ and $n = 4$.

Lemma 9. *Let* $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \Sigma_3$. *Then for some subset* $J \subseteq \{1, 2, 3\} : \sum\limits_{i\in J} \alpha_i = 1$ *or* $\alpha_1 + \alpha_2 + \alpha_3 = 2$. *To every pointed subset* J, *there corresponds a unique one-dimensional irreducible representation* π: $\pi(p_i) = 1$, $i \in J$, *and* $\pi(p_i) = 0$, $i \notin J$. *If* $\alpha_1 + \alpha_2 + \alpha_3 = 2$ *then, furthermore, the algebra has a unique, up to unitary equivalence, irreducible two-dimensional representation.*

Proof. The proof reduces to an easy computation, when taking into account that an irreducible pair of orthoprojections is a one-dimensionally or unitary equivalent to a pair

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \tau & \sqrt{\tau-\tau^2} \\ \sqrt{\tau-\tau^2} & 1-\tau \end{pmatrix}, \qquad 0<\tau<1. \qquad \blacksquare$$

Lemma 10. *If $\vec{\alpha} \in \Sigma_4^1$, $0 < A < 2$, is reduced then the following condition, which we will call the R-condition, is satisfied: $\exists\ J \subset \{1,2,3,4\} : \Sigma_{i\in J}\alpha_i = 1$ or $\exists\ \alpha_{i_0} : 2 - A = \alpha_{i_0}$.*

Proof. There are two possible cases.

1) Let $\pi(p_{i_0}) = 0$ then $\sum\limits_{i\neq i_0} \alpha_i \pi(p_i) = I$. Let $\vec{\alpha}'$ be obtained from $\vec{\alpha}$ by omitting the coordinate α_{i_0}. Obviously, $\vec{\alpha}' \in \Sigma_3$. So $\sum\limits_{i\in J} \alpha_i = 1$, for some subset $J \subset \{1,2,3,4\}\setminus\{i_0\}$, (if $\sum\limits_{i\neq i_0} \alpha_i = 2$, then $A > 2$).

2) If for all $\pi : \pi(p_{i_0}) = I$ then $\sum\limits_{i\neq i_0} \alpha_i \pi(p_i) = (1-\alpha_{i_0})I$. The operator at the left hand-side is positive. From here $\alpha_{i_0} \le 1$. If $\alpha_{i_0} = 1$, then the R-condition is satisfied, else $\sum\limits_{i\neq i_0} \frac{\alpha_i}{1-\alpha_{i_0}} \pi(p_i) = I$. From the previous lemma we have either: *a*) $\sum\limits_{i\in J} \frac{\alpha_i}{1-\alpha_4} = 1$, for some subset $J \subset \{1,2,3,4\}\setminus\{i_0\}$, hence $\sum\limits_{i\in J} \alpha_i + \alpha_4 = 1$ or *b*) $\frac{\alpha_1}{1-\alpha_4} + \frac{\alpha_2}{1-\alpha_4} + \frac{\alpha_3}{1-\alpha_4} = 2$, $\alpha_1+\alpha_2+\alpha_3 = 2(1-\alpha_4)$ and $2-A = \alpha_4$. $\blacksquare$

Note, that if $\vec{\alpha}$ satisfies R-condition then $\vec{\alpha}$ is not necessary reduced.

Lemma 11. *If $\vec{\alpha} \in \Sigma_4 \setminus \Sigma_4^1$ then $T(\vec{\alpha})$ satisfies R-condition.*

Proof. From the condition $\vec{\alpha} \in \Sigma_4 \setminus \Sigma_4^1$, we obtain $\alpha_{i_0} \ge 1$ for some i_0. Suppose $\alpha_{i_0} > 1$, $\pi \in \operatorname{Rep} \mathfrak{P}_{4,T(\vec{\alpha})}$ then, by the Lemma 3, $T(\pi)(p_{i_0}) = 0$. From here $\pi(p_{i_0}) = I$, so $\vec{\alpha}$ is reduced.

Assume $\alpha_{i_0} = 1$. From $T(\vec{\alpha}) = \left(\frac{\alpha_1}{A-1}, \frac{\alpha_2}{A-1}, \frac{\alpha_3}{A-1}, \frac{\alpha_4}{A-1}\right) = \left(\frac{\alpha_1}{\sum\limits_{i\neq i_0}\alpha_i}, \frac{\alpha_2}{\sum\limits_{i\neq i_0}\alpha_i}, \frac{\alpha_3}{\sum\limits_{i\neq i_0}\alpha_i}, \frac{\alpha_4}{\sum\limits_{i\neq i_0}\alpha_i}\right)$, the sum $\sum\limits_{j\neq i_0} \left(\frac{\alpha_j}{\sum\limits_{i\neq i_0}\alpha_i}\right) = 1$, so $T(\vec{\alpha})$ satisfies R-condition. $\blacksquare$

From Lemmas 2, 3, 8, 10, it follows

Lemma 12. *If $1 \le A < 1 + \frac{1}{n-2}\big|_{n=4} = \frac{3}{2}$ then $\vec{\alpha}$ satisfy R-condition.*

Using the lemmas proved above, we obtain:

Theorem 1. *Let $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $0 < \alpha_i < 1$, $A = \sum\limits_{i=1}^{4} \alpha_i$, Σ_4^1 be the set of such $\vec{\alpha}$ that the algebra $\mathfrak{P}_{4,\vec{\alpha}}$ has a nonzero representation.*

1) Dimensions of all irreducible representations of the algebra $\mathfrak{P}_{4,\vec{\alpha}}$ are finite.

2) If $A = 1$ then $\vec{\alpha} \in \Sigma_4^1$ and the corresponding algebra $\mathfrak{P}_{4,\vec{\alpha}}$ has a unique irreducible representation π, which is a one-dimensional representation and $\pi(p_i) = 1$.

3) If $A = 2$ then $\vec{\alpha} \in \Sigma_4^1$ and all irreducible representations has dimension one or two (their description see in [4]).

4) The functor S is equivalence of categories of representations of "symmetry" algebras $\mathfrak{P}_{4,(\alpha_1\alpha_2,\alpha_3,\alpha_4)}$ and $\mathfrak{P}_{4,(1-\alpha_1,1-\alpha_2,1-\alpha_3,1-\alpha_4)}$, $\vec{\alpha} \in \Sigma_4^1$, with the center of symmetry $A = 2$.

5) Every point $\vec{\alpha} \in \Sigma_4^1, 1 < A < 2$, or satisfies R-condition or $\Phi^-(\alpha)$ belongs to Σ_4^1.

6) $\vec{\alpha} \in \Sigma_4^1$, $1 < A < 2$ if and only if $\Phi^{-(k)}(\vec{\alpha})$ satisfy R-condition for some k.The number k is bounded by $N : \Phi^{-(N)}(A) \in [1, \frac{3}{2})$. The functor $\Phi^{-(k)}$ is equivalence of categories of representations of algebra $\mathfrak{P}_{n,\vec{\alpha}}$ and reduced algebra $\mathfrak{P}_{n,\Phi^{-(k)}(\vec{\alpha})}$.

The theorem allows us to reduce the solution of the problem about belonging of a point $\vec{\alpha}$ to Σ_4^1 to verifying R-condition for some another point.

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