

On the Spectral Problem for the Finite-Gap Schrödinger Operator

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Solving of the spectral problem for the finite-gap Schrödinger operator in terms of hyperelliptic Weierstrass functions is proposed. Corresponding solutions with help of unknown coefficients are expressed through the Weierstrass functions which also contain unknown parameters. These unknown quantities are determined by corresponding band equations and polynomial solutions of the inverse Jacobi problem. Corresponding equations can be reduced to simple algebraic equations. The elliptic finite-gap case is considered in the framework of the proposed approach.

1 Introduction

The spectral problem for finite-gap linear differential operator is interesting both of its own and as an auxiliary problem in the finite-gap theory of integrable partial differential equations. Furthermore it may be applied to electron spectra theory.

The spectral problem is reduced to building of finite-gap eigenfunctions and finding of their parameters from the spectral linear differential equation. Symmetrized products of the functional part of these parameters (so-called μ -functions) are expressed through functional coefficients (so-called potentials) of a linear differential operator by the fundamental system of finite-gap equations (see [1, 2]). This system follows from the comparison of asymptotic series developments of general and finite-gap eigenfunctions.

Usually, solving of the spectral linear differential equation in μ -functions is realized with help of the known Abelian transformation with a subsequent introducing of the corresponding Riemann surface. In so doing, μ -functions are considered as points of this surface and its symmetrized degrees are solutions of the Jacobi inversion problem (see [3, 4]). Corresponding solutions are expressed in terms of the Riemann theta functions.

Thus, the system of the finite-gap equations and the Abelian transformation leads to solving of the finite-gap spectral problem for linear differential operators through the Riemann theta functions. The above mentioned solution of the Jacobi problem is connected with the complicated analysis of properties of theta functions on the Riemann surface. At the same time utilization of the known (see [5, 6, 7]) relations for 2-differential of second kind can lead to essential simplification of the latter problem.

Taking above mentioned circumstances we suggest simplification for solving the spectral problem for finite-gap linear differential operators in the case of the Schrödinger operator in the class of hyperelliptic finite-gap functions. Consideration will be based on known relations for fundamental 2-differential on the hyperelliptic Riemann curves and the system of finite-gap equations connecting the hyperelliptic Weierstrass functions and its derivatives.

The paper is organized as follows. In Section 2 the building of the hyperelliptic finite-gap eigenfunction and finite-gap equations of the spectral problem for the Schrödinger operator are formulated. In Section 3 solving of the Jacobi inversion problem with the help of the known relations for the fundamental 2-differential on the Riemann hyperelliptic curves is considered.

In Section 4 on the basis of the finite-gap equation relation for the hyperelliptic Weierstrass functions are obtained.

2 The finite-gap function and finite gap equations for the Schrödinger operator

The general form of eigenfunctions for one-dimensional Schrödinger operator $H = -\partial_x^2 + U(x)$ (where $\partial_x^n \equiv d^n/dx^n$) is determined by the symmetry of the Schrödinger equation $H\Psi(x, E) = E\Psi(x, E)$, where $\Psi(x, E)$ means the eigenfunction, x and E are space and spectral variables respectively. Such symmetry is expressed in each specific case by corresponding integrals of motion. In the case under consideration when the differential equation has two solutions $\Psi_1(x, E)$ and $\Psi_2(x, E)$ this integral of motion has the form

$$\Psi_2(x, E)\partial_x\Psi_1(x, E) - \Psi_1(x, E)\partial_x\Psi_2(x, E) = 2G, \tag{1}$$

where G means a constant.

Introducing the variable $X(x, E) = \Psi_1(x, E)\Psi_2(x, E)$ we can write the evident relation

$$\Psi_2(x, E)\partial_x\Psi_1(x, E) + \Psi_1(x, E)\partial_x\Psi_2(x, E) = \partial_x X(x, E). \tag{2}$$

The system of two equations (1) and (2) result in the equation

$$\partial_x \ln \Psi_1(x, E) = \frac{12}{\partial_x} \ln X(x, E) + \frac{G}{X(x, E)}. \tag{3}$$

The solutions of this equation

$$\Psi_{1,2}(x, E) = \sqrt{X(x, E)} \exp\left(\pm \int_{x_0}^x dx \frac{G}{X(x, E)}\right) \tag{4}$$

determine the general form of the Schrödinger eigenfunctions taking into account the symmetry of the system (see [8]).

The finite-gap case imposes on the X -function the polynomial dependence on E . Then the differentiation (3) with taking into account this circumstance results in the equality (see [8])

$$U(x) = \frac{1}{2X} \partial_x^2 - \frac{1}{2X^2} (\partial_x X)^2 + \left(\frac{G}{X}\right).$$

Multiplication of the last on X^2 at zero points $x = a_i$ ($X(a_i) = 0$) yields the relation

$$\partial_x X|_{x=a_i} = 4G. \tag{5}$$

This relation is used for computation of the finite-gap eigenfunctions.

The above mentioned general Schrödinger eigenfunction in terms of the function $\chi = G/X$ can be written in the form

$$\Psi(x, E) = \sqrt{\chi(x, E)} \exp\left(\int^x dx \chi(x, E)\right), \tag{6}$$

where χ is real function with the asymptotic series

$$\chi(x, E) = \sqrt{E} \left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \chi_{2n+1}(x) E^{-(n+1)}\right) \tag{7}$$

(further we shall omit the argument of the coefficient functions χ_n). Coefficients of (7) satisfy the known [1] recurrence relation

$$\chi_{n+1} = \frac{d}{dx}\chi_n + \sum_{k=1}^{n-1} \chi_k \chi_{n-k}, \quad \chi_1 = -U(x), \tag{8}$$

from which follow that χ_n -functions are polynomial in the potential U and its derivatives.

Thus power series (7) and the recurrence relation (8) determine the power series of χ -function in the expression (4) for the general eigenfunction through the Schrödinger potential and its derivatives.

The finite-gap spectrum of the Schrödinger operator imposes the condition of a polynomial form of the X -function in the expression (4). In the case of g -gap spectra (which have g gaps and $2g + 1$ boundaries $\{E_i\}$) the quantities G and X are described by the expressions [1]

$$G = \sqrt{P(E)} = \sqrt{\prod_{n=1}^{2g+1} (E - E_n)}, \quad X = Q(E, x) = \prod_{n=1}^g (E - \mu_n(x)). \tag{9}$$

Then the χ -function transforms to the form

$$\chi_R(x, E) = \frac{\sqrt{\sum_{n=0}^{2g+1} a_n E^{-n}}}{\sum_{n=0}^g b_n E^{-n}}, \quad a_0 = 1, \quad b_0 = 1. \tag{10}$$

Here a_n and b_n are symmetrized products of spectral boundaries E_j and μ -functions of the n th order, respectively;

$$a_n = (-1)^n \sum_{j_1, \neq j_2, \dots, \neq j_n}^{2g+1} \prod_{i=1}^n E_{j_i}, \quad b_n = (-1)^n \sum_{j_1, \neq j_2, \dots, \neq j_n}^g \prod_{i=1}^n \mu(x)_{j_i},$$

The expression (10) as in the case (7) can be represented in the form of the asymptotic series

$$\chi_R \sim \sqrt{E} \left(1 + \sum_{n=1}^{\infty} A_n E^{-n} \right), \tag{11}$$

with coefficients

$$A_n = \frac{1}{n!} \frac{d^n}{dz^n} \sqrt{\frac{\sum_{n=0}^{2g+1} a_n z^n}{\sum_{n=0}^g b_n z^n}} \Big|_{z=0}. \tag{12}$$

Comparing coefficients at the same power of E^{-1} in the expressions (7) and (11) we obtain finite-gap equations

$$\frac{(-1)^n}{2^{2n+1}} \chi_{2n+1} = \frac{1}{(n+1)!} \partial_z^{n+1} \left(\frac{\sqrt{\sum_{n=0}^{2g+1} a_n z^n}}{\sum_{n=0}^g b_n z^n} \right) \Big|_{z=0}, \quad b_0 = 0. \tag{13}$$

The system (13) in accord with the definition (12) and (8) determines relations between coefficient functions b_n (symmetrized products of μ_i -functions), a_n and polynomials of n th power in the potential $U(x)$ and its derivatives. The first g equations of this (13) are a system of algebraic equations which is solvable in respect to (b_1, \dots, b_g) . This system at $n \geq g + 1$ determines relations between the Schrödinger potential and its derivatives.

Thus finite-gap Schrödinger eigenfunctions and potentials are determined by the finite-band equations (13) presenting by relations between symmetrized products of μ -functions (b_i -coefficient functions) and the Schrödinger potential (U) with its derivatives. But complete solution of the spectral problem assumes computation of symmetrized products of μ -functions.

The general hyperelliptic finite-gap Schrödinger operator U in accord with the finite-band equation (13) is described linear symmetrized combination of μ -functions by the expression

$$U(x) = 2 \sum_{j=1}^g \mu_j(x) - \sum_{j=1}^{2g+1} E_j. \tag{14}$$

The linear combination of μ -functions in (14) can be obtained by substitution the above mentioned finite-gap function in the Schrödinger equation and using the Abelian change of variables. In so doing, the solution is reduced to the Jacobi inverse problem. The latter is solving with help of a theorem about theta function zeros.

3 Calculating the symmetrized products of μ -functions

Substitution of the finite-gap Schrödinger eigenfunction (4) taking into account (9) in the Schrödinger equation results in the differential equation with respect to μ_i -functions. In the class of the Abelian hyperelliptic functions it can be integrated by the Abelian transformation of the form

$$\mathbf{v} = \sum_{j=1}^g \int_{P_0}^{\mu_j(z)} d\mathbf{v}, \quad d\mathbf{v} = (2\omega)^{-1} d\mathbf{u}, \quad du_j = \frac{z^{j-1}}{y(z)}, \tag{15}$$

Here $y^2(z) = \sum_{j=1}^{2g+1} \lambda_j z^j$ means a hyperelliptic Riemann curve Γ ; du_j means holomorphic differential of the first kind on Γ . Moreover, 2ω means a matrix of periods on the canonical basis cycles a_j (see [9]) on the Riemann surface Γ ,

$$(2\omega)_{i,j=1,\dots,g} = \left(\oint_{\mathbf{a}_i} du_j \right),$$

which exists together with a matrix $2\omega'$ of periods on the canonical basis cycles b_j ,

$$(2\omega')_{i,j=1,\dots,g} = \left(\oint_{\mathbf{b}_i} du_j \right).$$

The first matrix of periods on the Riemann surface is obtained from the condition reducing the canonical basis of holomorphic differentials du_j to the normal form dv_j .

Thus the finite-gap spectral problem for the Schrödinger operator was reduced to the problem of the Abelian integral (15) conversion with respect to the symmetrized products of μ_i -functions, i.e. the Jacobi inversion problem.

Taking into account (14) and using the known Riemann vanishing theorem for theta Riemann function $\theta(z|\tau)$ (which will be defined below) we can obtain the expression

$$U(x) = 2 \sum_{i,j} \alpha_i \alpha_j \partial_{\alpha_i, \alpha_j} \ln \theta(\alpha x - K|\tau), \tag{16}$$

(where $\alpha_i = (2\omega)_{gi}^{-1}$) for hyperelliptic U -potentials.

Calculation of symmetrized products of higher degree can be realized with help of the so-called fundamental 2-differential of the second kind which is defined through the function of the form (see [7, 9])

$$F(z_1, z_2) = 2y_2^2 + 2(z_1 - z_2)y_2\partial_z y_2 + (z_1 - z_2)^2 \sum_{j=1}^g z_1^{j-1} \sum_{k=j}^{2g+1-j} (k - j + 1)\lambda_{k+j+1}z_2^k, \quad (17)$$

$$F(z_1, z_2) = 2\lambda_{2g+2}z_1^{g+1}z_2^{g+1} + \sum_{i=0}^g z_1^i z_2^i (2\lambda_{2i} + \lambda_{2i+1}(z_1 + z_2)). \quad (18)$$

Here any pair of points $(y_1, z_1), (y_2, z_2) \in \Gamma$.

Then the fundamental Abelian 2-differential of the second kind with the unique pole of the second order along $z_1 = z_2$ can be written in the form

$$d\hat{\omega}(z_1, z_2) = \frac{2y_1y_2 + F(z_1, z_2)}{4(z_1 - z_2)^2} \frac{dz_1}{y_1} \frac{dz_2}{y_2}. \quad (19)$$

Taking into account (17) the expression (19) can be rewritten in the form

$$d\hat{\omega}(z_1, z_2) = \frac{\partial}{\partial z_2} \left(\frac{y_1 + y_2}{2y_1(z_1 - z_2)} \right) dz_1 dz_2 + d\mathbf{u}^T(x_1) d\mathbf{r}(x_2), \quad (20)$$

where

$$dr_j = \sum_{k=j}^{2g+1-j} (k + 1 - j)\lambda_{k+1+j} \frac{z^k dz}{4y}, \quad j = 1, \dots, g \quad (21)$$

is a canonical Abelian differential of the second kind.

Solution of the Jacobi inversion problem (15) is based on the known relation of the fundamental 2-differential (19) which can be written as

$$\int_{\mu}^z \sum_{i=1}^g \int_{\mu_i}^{z_i} \frac{2yy_i + F(z, z_i)}{4(z - z_i)^2} \frac{dz}{y} \frac{dz_i}{y_i} = \ln \left\{ \frac{\theta \left(\int_{a_0}^z d\mathbf{v} - \sum_{i=1}^g \int_{a_i}^{z_i} d\mathbf{v} \right)}{\theta \left(\int_{a_0}^z d\mathbf{v} - \sum_{i=1}^g \int_{a_i}^{\mu_i} d\mathbf{v} \right)} \right\} - \ln \left\{ \frac{\theta \left(\int_{a_0}^{\mu} d\mathbf{v} - \sum_{i=1}^g \int_{a_i}^{z_i} d\mathbf{v} \right)}{\theta \left(\int_{a_0}^{\mu} d\mathbf{v} - \sum_{i=1}^g \int_{a_i}^{\mu_i} d\mathbf{v} \right)} \right\}. \quad (22)$$

The right hand of this equation contains the known Riemann theta function (see [1])

$$\mathcal{R}(z) = \theta(\mathbf{w}(z)|\tau) = \sum_{\mathbf{m} \in Z^n} \exp \{ \pi i (\mathbf{m}^T \tau \mathbf{m}) + 2\pi i (\mathbf{w}(z)^T \mathbf{m}) \}. \quad (23)$$

Here $(a \cdot b)$ means a scalar product,

$$\mathbf{w}(z) = \int_{a_0}^z d\mathbf{v} + \sum_{k=1}^g \int_{z_0}^{z_k} d\mathbf{v} - \mathbf{K}_{z_0},$$

where components of \mathbf{K} defined as

$$K_j = \frac{1 + \tau_{jj}}{2} - \sum_{l \neq j} \oint_{a_l} d\mathbf{v}_l(x) \int_{z_0}^z d\mathbf{v}_j, \quad j = 1, \dots, g$$

is the vector of Riemann constants with respect to the base point z_0 . In the considered case with the base point a the vector of Riemann constants has the form $\mathbf{K}_a = \sum_{k=1}^g \int_a^{a_i} d\mathbf{v}$. Taking into account definition of the hyperelliptic Weierstrass function through theta function (23) as

$$\wp_{ij} = \partial_{v_i, v_j}^2 \ln \theta(v|\tau)$$

and differentiating (22) on variables z and z_r we can obtain the relation

$$\sum_{i=1}^g \wp_{ij} \left(\int_{a_0}^z d\mathbf{v} + \sum_{k=1}^g \int_{a_k}^{z_k} d\mathbf{v} + \mathbf{K}_a \right) z^{i-1} z_r^{j-1} = \frac{F(z, z_r) - 2yy_r}{4(z - z_r)^2}, \tag{24}$$

which expresses a second kind 2-differential through the linear combination of the hyperelliptic Weierstrass functions. In the limit case $z \rightarrow \infty$ from (24) the relation follows

$$\mathcal{P}(z; \mathbf{v}) = z^g - \sum_{j=1}^{g-1} \left(\sum_{l,m} \wp_{l,m}(\mathbf{v}) \alpha_l \alpha_j \right) z^{j-1}, \tag{25}$$

in which points $\{z_i\}$ of the Riemann surface are presented as roots of the polynomial \mathcal{P} ($z_j \equiv \mu_j$). In so doing, symmetrized products of these points are expressed by linear combinations hyperelliptic Weierstrass functions through the period matrix of holomorphic differentials 2ω . The α -coefficients are obtained from algebraic equations which can be obtained by substitution (16) in the finite-band equation (13) taking into account (25). One will be demonstrated in next section in the case elliptic finite-gap Schrödinger potentials.

The matrix 2ω can be calculated with help the known Thomae formulae of the form

$$\begin{aligned} \theta^4[\varepsilon(\mathbf{I}_0)] &= \pm \det 2\omega \prod_{i,j \in \mathbf{J}_0} (E_i - E_j) \prod_{n,m \in \tilde{\mathbf{J}}_0} (E_n - E_m), \\ \theta_j^4[\varepsilon(\mathbf{I}_1)] &= \pm \frac{\det(2\omega)^{-2}}{16} \prod_{i,j \in \mathbf{J}_1} (E_i - E_j) \prod_{n,m \in \tilde{\mathbf{J}}_1} (E_n - E_m) \sum_{i=1}^g \oint_{\mathbf{a}_i} \frac{z^{j-1} dz}{y} S_{i-1}(\mathbf{I}_1). \end{aligned}$$

Here expressions

$$S_0(\mathbf{I}_1) = 1, \quad S_1(\mathbf{I}_1) = \sum_{j \in \mathbf{I}_1} E_j, \quad \dots, \quad S_{g-1}(\mathbf{I}_1) = \prod_{j \in \mathbf{I}_1} E_j$$

denotes symmetrized products of the branching points of the Riemann surface.

Thus finite-gap Schrödinger eigenfunctions and potentials can be expressed through hyperelliptic Weierstrass functions containing theta-constants instead unknown elements of a 2ω -matrix.

4 The finite-gap relations for elliptic Weierstrass functions

The system of finite-gap equations (10) is solvable with respect to symmetrized products μ_i -functions expressed as coefficient functions b_i . Therefore excluding b_i -functions at $n > g$ we can obtain the system of algebraic equations with respect to the Schrödinger potential and its derivatives. These equations determine relations for the hyperelliptic Weierstrass functions.

We consider above mentioned relations in the case of the Riemann curves of low genus g which correspond to small number of gaps in the eigenvalue spectrum of the Schrödinger operator.

One-gap spectrum. The Schrödinger potential $U(z)$ is determined by the system of three finite-gap equations of the form (13) at $n = \overline{0, 2}$. Substitution into these equations of the explicit

expressions (12) for A_n and polynomial in U expressions for χ_n -functions which follow from (8) yields the system

$$\begin{aligned} \frac{1}{2}a_1 - b_1 &= -\frac{1}{2}U, & \frac{1}{2} \left\{ \left(a_2 - \frac{1}{4}a_1^2 \right) + 2(b_1^2 - b_2) - a_1b_1 \right\} &= -\frac{1}{2^3} \left\{ U^2 - U^{(2)} \right\}, \\ \frac{1}{3!} \left\{ \left(\frac{3}{8}a_1^3 - \frac{3}{2}a_1a_2 + \frac{3}{2}a_3 \right) + 3 \left(\frac{1}{4}a_1^2 - a_2 \right) b_1 + 3a_1(b_1^2 - b_2) + (12b_1b_2 - 4!b_3 - 6b_1^3) \right\} \\ &= \frac{1}{2^5} \left\{ U^{(4)} - 5U^{(1)^2} + 6UU^{(2)} - 2U^3 \right\} \end{aligned}$$

in which $b_n|_{n \geq 2} = 0$ (in view of the relation $b_n|_{n \geq g+1} = 0$, where g is the number of gaps in the eigenvalue spectrum of the Schrödinger equation. Excluding b_n from the last system we can obtain the equations

$$\begin{aligned} b_2 = 0 &= \frac{1}{8} \left(3U^2 - U^{(2)} \right) + \frac{1}{4}a_1U + \frac{1}{2}a_2 - \frac{1}{8}a_1^2, \\ b_3 = 0 &= -\frac{1}{32} \left(U^{(4)} + 10U^3 - 5U^{(1)^2} - 10UU^{(2)} \right) - \frac{1}{16}a_1 \left(3U^2 - U^{(2)} \right) \\ &+ \frac{1}{16}U \left(a_1^2 - 4a_2 \right) + \frac{1}{2}a_3 + \frac{1}{4}a_1a_2 - \frac{1}{16}a_1^3. \end{aligned}$$

Inserting into the latter system the expression $U = 2\alpha_1^2\wp_{1,1} - a_1$ (in accord with (25)) we can obtain unknown parameters for 1-gap Schrödinger potentials.

Two-gap spectrum. The 2-gap Schrödinger potential is determined by the system of the four finite-gap equations of the form (13) at $n = \overline{0,3}$. Analogically to the one-gap case their explicit form can be obtained by the substitution of the expressions (12) for A_n and expressions for χ_n (following from (8)) into (13). In so doing, the first two equations are solvable with respect to b_1 and b_2 . Excluding the latter from the fourth and fifth equation and taking into account the equality $b_n|_{n \geq 3} = 0$ we can obtain the finite-gap system

$$\begin{aligned} b_3 = 0 &= \frac{1}{2^5} \left(16a_3 + 8a_2U + 10U^3 - 5U'^2 - 2a_1U'' - 10UU'' + U^{(4)} \right), \\ b_4 = 0 &= \frac{1}{2^7} \left(-16a_2^2 + 64 * a_4 + 32a_3U + 24a_2U^2 + 35U^4 \right. \\ &\left. - 70UU'^2 - 8a_2U'' - 70U^2U'' + 21U''^2 + 28U'U^{(3)} + 14UU^{(4)} - U^{(6)} \right). \end{aligned}$$

Inserting into the latter system the expression $U = 2 \sum_{i,j} \alpha_i \alpha_j \wp_{i,j} - a_1$, $i, j = 1, 2$ (in accord with (25)) we can obtain unknown parameters for 2-gap Schrödinger potentials.

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