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Symmetry of Nonlinear Schrödinger Equations with Harmonic Oscillator Type Potential

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The group classification in the class of nonlinear Schrödinger equations of the form $i\,\psi_t + \Delta\psi + k|x|^2\psi - f\,(|\psi|)\,\psi = 0$ was carried out. The maximal Lie invariance algebra of such equations was calculated.

The study of nonlinear Schrödinger equations using symmetry methods began in 1972 by Niederer's article [1], in which maximal Lie invariance algebra (MIA) of the free Schrödinger equation was calculated for the first time. In 1973 U. Niederer [2] calculated the MIA of the linear Schrödinger equation with the harmonic oscillator type potential. In article [4] nonlinear Schrödinger equations of the form

$$i\,\psi_t + \Delta\psi + F\,(t, x, \psi, \psi^*, \psi_t, \psi_t^*) = 0$$

are considered, and classification of one-dimensional equations which admit the MIA of dimension n ($n \le 3$) is carried out.

In this article the group classification of the nonlinear generalized Schrödinger equations

$$i\,\psi_t + \Delta\psi + k|x|^2\psi - f\,(|\psi|)\,\psi = 0 \tag{1}$$

in the n-dimensional space is performed. The differentiable function $f = f(|\psi|)$ and the constant k are arbitrary elements. As a particular case of the problem the invariance algebra of the free Schrödinger equations and the Schrödinger equations with the harmonic oscillator type potential are found. The linear equation (the case $f = 0$) was considered in papers [1, 2] and given here for the completeness of results only. The results of this article for $k = 0$ coincide with results of paper [3], which is devoted to group classification of the nonlinear Schrödinger equations of the form

$$i\,\psi_t + \Delta\psi + F\,(\psi, \psi^*) = 0.$$

Theorem 1. *The Lie algebra of the kernel of principal groups of equation (1) is*

$$A^{\ker} = \langle \partial_t, J_{ab}, M \rangle.$$

The Lie algebra of the kernel of principal groups of equation (1) for fixed k is

1) $A_0^{\ker} = \langle \partial_t, \partial_a, J_{ab}, G, M \rangle$ *in the case* $k = 0$;
2) $A_-^{\ker} = \langle M, \partial_t, J_{ab}, e^{2\varkappa t}(\partial_a + \varkappa x^a M), e^{-2\varkappa t}(\partial_a - \varkappa x^a M) \rangle$
   in the case $k = -\varkappa^2$, $\varkappa > 0$;
3) $A_+^{\ker} = \langle M, \partial_t, J_{ab}, \sin 2\varkappa t\,\partial_a + \varkappa x^a \cos 2\varkappa t\, M, \cos 2\varkappa t\,\partial_a - \varkappa x^a \sin 2\varkappa t\, M \rangle$
   in the case $k = \varkappa^2$, $\varkappa > 0$.

Here $M = i\,(\psi\,\partial_\psi - \psi^*\partial_{\psi^*})$, $G = t\partial_a - \frac{x^a}{2}\,M$, $J_{ab} = x^b\partial_a - x^a\partial_b$.

Theorem 2. *The class of equations (1) admits the following equivalence transformations:*

$$\begin{aligned}
&1)\ \tilde t = t,\ \tilde x = x,\ \tilde\psi = e^{i\alpha t}\psi,\ \tilde f = f - \alpha,\ \tilde k = k;\\
&2)\ \tilde t = \varepsilon^2 t,\ \tilde x = \varepsilon x,\ \tilde\psi = \psi,\ \tilde f = \varepsilon^{-2} f,\ \tilde k = \varepsilon^{-4} k;\\
&3)\ \tilde t = t,\ \tilde x = x,\ \tilde\psi = \varepsilon\psi,\ \tilde f = f,\ \tilde k = k.
\end{aligned} \tag{2}$$

Here $\alpha, \varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$.

The classification of extension of the MIA will be carried out accurate to transformations (2).

Theorem 3. *The complete set of nonequivalent cases of extension of the MIA of equations (1) are exhausted by the following (we adduce only operators from extensions of algebra* $A_0^{\ker}$, $A_-^{\ker}$, $A_+^{\ker}$ *for the cases* $k = 0$, $k = -\varkappa^2$, $k = \varkappa^2$ *correspondingly)*

$$\begin{array}{ll}
1)\ f = (\delta_1 + i\delta_2)|\psi|^\gamma,\ \gamma \neq 0, 4/n: & I - \gamma D;\\
2)\ f = (\delta_1 + i\delta_2)|\psi|^{4/n}: & I - \dfrac{4}{n} D,\ \Pi;\\
3)\ f = i\delta_2 \ln|\psi|: & I + \delta_2 t M;\\
4)\ f = (\delta_1 + i\delta_2)\ln|\psi|,\ \delta_1 \neq 0: & e^{\delta_1 t}(\delta_1 I + \delta_2 M);\\
5)\ f = 0,\ k = 0: & I,\ D,\ \Pi;\\
6)\ f = 0,\ k = -\varkappa^2,\ \varkappa > 0: & I,\ e^{4\varkappa t}\left(\partial_t + 2\varkappa x^a\partial_a + 4\varkappa^2|x|^2 M - n\varkappa I\right),\\
 & e^{-4\varkappa t}\left(\partial_t - 2\varkappa x^a\partial_a + 4\varkappa^2|x|^2 M + n\varkappa I\right);\\
7)\ f = 0,\ k = \varkappa^2,\ \varkappa > 0: & I,\ \cos 4\varkappa t\left(\partial_t + 2\varkappa^2|x|^2 M\right) - \sin 4\varkappa t\,(2\varkappa x^a\partial_a - \varkappa n I),\\
 & \sin 4\varkappa t\left(\partial_t + 2\varkappa^2|x|^2 M\right) + \cos 4\varkappa t\,(2\varkappa x^a\partial_a - \varkappa n I).
\end{array}$$

Here $I = \psi\partial_\psi + \psi^*\partial\psi^*$, $D = 2t\partial_t + x^a\partial_a - \frac{n}{2}I$, $\Pi = t^2\partial_t + tx^a\partial_a + \frac{|x|^2}{4}M - \frac{nt}{2}I$, $\{\delta_1,\ \delta_2,\ \gamma\} \subset \mathbb{R}$.

The results of group classification of generalized nonlinear Schrödinger equations obtained in this article can be used for the construction of exact solutions of these equations. These results can be considered as the basis for further analysis of generalized nonlinear Schrödinger equations. We plan to finish complete group classification for the case $f = f(\psi, \psi^*)$ and an arbitrary potential $V = V(t, x)$ and to construct exact solutions of such generalized nonlinear Schrödinger equations.

Because of the extensions of the MIA in the cases $k = 0$ and $k \neq 0$ are equal we hope to build the equivalence transformations between these classes.

We are going to use the results of present paper for the investigation of Q-conditional (non-classical) symmetries of Schrödinger equations.

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