Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 1, 321–322

On Integrability of Some Nonlinear Model with Variable Separant

Yelyzaveta HVOZDOVA

Lviv Commercial Akademy, Lviv, Ukraine
E-mail: *matmod@franko.lviv.ua*

In this paper a new integrable nonlinear Hamiltonian system in $(1+1)$-dimension is introduced. Nontrivial connection with well-known multicomponent nonlinear Schrödinger model is found.

Let us consider a non-linear Hamiltonian system

$$\psi_t = \{\psi, H\} \tag{1}$$

in the Schwarz space of smooth fast decreasing on the $\pm\infty$ complex value l-component vector-functions $\psi = (\psi_1, \ldots, \psi_l)(x)$, $l \in \mathbb{N}$ of the variable $x \in \mathbb{R}$ with the Hamiltonian

$$H = \int_{-\infty}^{+\infty} |\psi_x|^2 dx, \tag{2}$$

and local brackets of Poisson for dynamic variables ψ_m, ψ_n, $m, n = \overline{1, l}$:

$$\left\{\psi_n(x), \bar{\psi}_m(y)\right\} := i\delta_n^m \left(c + |\psi|^2(x)\right)^2 \delta(x-y), \tag{3}$$

where δ_n^m is the Kronecker symbol, $\delta(z)$ is the Dirac function, $c \in \mathbb{R}$.

System (1)is non-linear evolutionary system of differential equations with variable separant (coefficient at higher derivative) and has the next form:

$$i\psi_t = -\left(c + |\psi|^2\right)^2 \frac{\delta H}{\delta \psi^*} = \left(c + |\psi|^2\right)^2 \psi_{xx}, \tag{4}$$

where $\frac{\delta}{\delta\psi^*}$ is the Euler operator of variative derivative over the vector-function $\psi^* := \bar{\psi}^\top$.

Proposition 1. *Hamiltonian system (1)–(4) is formally integrable (by Lax) and assumes infinitive hierarchy non-trivial local laws of motion.*

Proof. For simplicity we restrict ourselves with Lax commutative representation discovered by us $[L, M] := LM - ML = 0$ in algebra of integro-differential operators [1, 2] which is equivalent to system (4), where

$$L = \left(c + |\psi|^2\right)\mathcal{D} + \psi_x\psi^* - \psi_x\mathcal{D}^{-1}\psi_x^*, \tag{5}$$

$$M = i\partial_t - \left(c + |\psi|^2\right)^2 \mathcal{D}^2 - 2\left(c + |\psi|^2\right)|\psi|_x^2\mathcal{D} = i\partial_t - \left(L^2\right)_{>0}, \tag{6}$$

and, as consequence of operators commutativity in (5)–(6), known [1] procedure for finding density ρ_k of first integrals $H_k := \int_{-\infty}^{+\infty} \rho_k dx$:

$$\rho_k = \operatorname{Res}\left(L^k\right), \qquad k \in \mathbb{Z}. \tag{7}$$

■

Remark 1. Obviously, $k = 1$ corresponds to Hamiltonian $H(2)$, and one of the simplest first integrals ($k = -1$) in the formula (7) has the form:

$$H_{-1} = \int_{-\infty}^{+\infty} \frac{|\psi|^2}{c + |\psi|^2} dx, \qquad c \in \mathbb{R} \setminus \{0\}.$$

Remark 2. In the formula (5) integral item $\psi_x \mathcal{D}^{-1} \psi_x^*$ is a symbol of skew-Hermitian operator of Volterra $\widehat{V}$ with the degenerated kernel $V(x, s) := \frac{\partial \psi(x)}{\partial x} \frac{\partial \psi^*(s)}{\partial s}$

$$\left(\widehat{V} f\right)(x) = \frac{1}{2} \left\{ \int_{-\infty}^{x} \sum_{i=1}^{l} \frac{\partial \psi_i(x)}{\partial x} \frac{\partial \bar{\psi}_i(s)}{\partial s} f(s) ds - \int_{x}^{+\infty} \sum_{i=1}^{l} \frac{\partial \psi_i(x)}{\partial x} \frac{\partial \bar{\psi}_i(s)}{\partial s} f(s) ds \right\}.$$

The symbol $\left(L^k\right)_{>0}$ strands for the differential part without free term (multiplier operator by function) of an integro-differential operator L^k.

Proposition 2. *The followng non-local replacement of variables* $(t, x, \psi) \to (\tau, y, \varphi)$:

$$\tau = t, \qquad y'_x = \frac{1}{c + |\psi|^2}, \qquad \varphi(\tau, y) = \frac{\psi_y}{c + |\psi|^2} \exp \int_{-\infty}^{y} \frac{\psi_y \psi^*}{c + |\psi|^2} dy \tag{8}$$

transforms non-linear system (4) into the multicomponent non-linear equation of Schrödinger [3]

$$i\varphi_\tau = \varphi_{xx} + 2|\varphi|^2 \varphi. \tag{9}$$

Proof. The proof is conducted by direct calculation. We restrict ourselves by the Lax operator (5). Making replacement (8) we get

$$L = \left(c + |\psi|^2\right) \mathcal{D}_x + \psi_x \psi^* - \psi_x \mathcal{D}_x^{-1} \psi^* \to \widetilde{L} = \mathcal{D}_y + \frac{\psi_y \psi^*}{c + |\psi|^2} - \frac{\psi_y}{c + |\psi|^2} \mathcal{D}_y^{-1} \psi_y^*,$$

and after gauge trasformation $\widetilde{L} \to \Phi \widetilde{L} \Phi^{-1}$ with the function $\Phi = \exp \int_{-\infty}^{y} \frac{\psi_y \psi^*}{c + |\psi|^2} dy$ the operator L to pass into the Lax operator L_{NS} [2, 4, 5] for the model (9):

$$L_{NS} = \Phi \widetilde{L} \Phi^{-1} = \mathcal{D}_y - \varphi \mathcal{D}^{-1} \varphi^*,$$

where the dynamic variable $\varphi = \varphi(\tau, y)$ is defined by substitution (8). ■

[1] Dickii L.A., Soliton equations and Hamiltonian systems, *Adv. Ser. Math. Phys.*, 1991, V.12.

[2] Samoilenko A.M., Samoilenko V.G. and Sidorenko Yu.M., Hierarchy of the Kadomtsev–Petviashvili equations under nonlocal constraints: many-dimensional generalizations and exact solutions of reduced system, *Ukr. Math. J.*, 1999, V.51, N 1, 78–97.

[3] Manakov S.V., A remark on the integrability of the Euler equation of dynamics of an n-dimensional solid body, *Funkts. Anal. Prilozhen.*, 1976, V.10, N 4, 93–94.

[4] Sidorenko Yu. and Strampp W., Multicomponents integrable reductions in Kadomtsev–Petviashvili hierarchy, *J. Math. Phys.*, 1993, V.34, N 4, 1429–1446.

[5] Oevel W., Sidorenko Yu. and Strampp W., Hamiltonian structures of the Melnikov system and its reductions, *Inverse Probl.*, 1993, V.9, 737–747.