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An Old Problem Newly Treated with Differential Forms: When and How Can the Equation $y'' = f(x, y, y')$ Be Linearized?

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Sophus Lie, more than a century ago, investigated the problem of linearization of the equation $y'' = f(x, y, y')$, where $()'$ means d/dx [1]. Originally, he investigated the necessary conditions for linearization by a point transformation and showed that f must be a cubic in y' and that other conditions must be satisfied. Later, he and others such as Tresse [2] worked out actual construction of the linearizing transformations, often using group theory. The present author will show a method of construction using differential forms, suitable when certain intermediate equations can be integrated explicitly.

1 Introduction

The possible linearization of the equation

$$y'' = f(x, y, y'), \tag{1}$$

where the prime indicates differentiation with respect to x, might be considered a simple problem, but it is actually rather complex. It is a very old problem, having been investigated by Sophus Lie [1] and by other subsequent authors (for example Tresse [2], Ibragimov [3, 4, 5], Berkovich [6], Grissom et al [7], Kamran et al [8, 9], Bocharov et al [10], Schwarz [11], Steeb [12], and N. Euler [13]). These methods use group theory or approach the problem as a Cartan equivalence problem. There are also treatments that consider equivalence of nonlinear and linear partial differential equations, such as those by Kumei and Bluman [14, 15].

In 1998 the author spent a month at the University of Witwatersrand in South Africa as the guest of Fazal Mahomed. During that month this was one of the problems that we looked at, and it became intriguing as the possibility of using differential forms in its treatment emerged. This talk is a detailed report on that research. The author has reported on it before in a summary fashion [16].

The previous papers that treat this as a Cartan equivalence problem use the Cartan theory. The differential forms used here are not part of that theory, but are used to make the treatment simpler and more obvious. One can carry out the same calculations without forms. The virtue of this approach is that the linearization can be achieved, in principle, by solving some intermediate linear differential equations. We will see how this can be done.

2 Basic theory and conditions for linearizability

We begin by adopting Lie's approach: assume a point transformation given by new variables

$$X = F(x, y), \qquad Y = G(x, y), \tag{2}$$

and require that

$$d^2Y/dX^2 = 0. \tag{3}$$

We note that this is a special case of a linear equation. Lie does not consider other cases, with terms in dY/dX and Y. We will comment on this later.

Now consider the conditions imposed on $F(x, y)$ and $G(x, y)$ by this requirement. We first construct, using equation (2),

$$dY/dX = (G_x dx + G_y dy)/(F_x dx + F_y dy) = \left(G_x + G_y y'\right) / \left(F_x + F_y y'\right),$$

where subscripts x and y denote differentiation. Now the second derivative equation may be written simply in terms of a differential $d(dY/dX) = 0$, or

$$\left(F_x + F_y y'\right) d\left(G_x + G_y y'\right) - \left(G_x + G_y y'\right) d\left(F_x + F_y y'\right) = 0,$$

which now may be treated as a differential form equation. We expand the differentials and obtain

$$\left(F_x + F_y y'\right)\left(dG_x + y' dG_y + G_y dy'\right) - \left(G_x + G_y y'\right)\left(dF_x + y' dF_y + F_y dy'\right) = 0$$

or

$$T dy' + \rho {y'}^2 + (\lambda + \delta) y' + \sigma = 0, \tag{4}$$

where

$$T = F_x G_y - F_y G_x$$

and we have the $1-$forms

$$\begin{aligned} \rho &= F_y dG_y - G_y dF_y, & \lambda &= F_y dG_x - G_y dF_x, \\ \sigma &= F_x dG_x - G_x dF_x, & \delta &= F_x dG_y - G_x dF_y. \end{aligned} \tag{5}$$

We note that

$$dT = \delta - \lambda. \tag{6}$$

Rewrite equation (4) as

$$dy' = \alpha + \beta y' + \gamma {y'}^2, \tag{7}$$

where

$$\alpha = -\sigma/T, \qquad \beta = -(\lambda + \delta)/T, \qquad \gamma = -\rho/T. \tag{8}$$

This sort of equation has occurred in other contexts, such as in searching for Bäcklund transformations, where y' may be viewed as a fiber coordinate on a base space parameterized by x and y.

We remember from differential form calculus that $dd\omega = 0$, where ω is any form, and that $1-$forms anticommute under the hook product $\wedge$. For integrability of equation (7), we ask $ddy' = 0$, or

$$0 = d\alpha + dy' \wedge \beta + y' d\beta + 2y' dy' \wedge \gamma + {y'}^2 d\gamma,$$

and with substitution from equation (7) we have

$$0 = d\alpha + \left(\alpha + \beta y' + \gamma {y'}^2\right) \wedge \beta + y' d\beta + 2y' \left(\alpha + \beta y' + \gamma {y'}^2\right) \wedge \gamma + {y'}^2 d\gamma.$$

The y'^3 term vanishes because $\gamma \wedge \gamma = 0$; we equate the coefficients of the other powers of y' to zero and get

$$d\alpha = \beta \wedge \alpha, \qquad d\beta = 2\gamma \wedge \alpha, \qquad d\gamma = \gamma \wedge \beta. \tag{9}$$

Now we go back to equations (5) and expand the differentials:

$$\begin{aligned}
\rho &= F_y(G_{xy}dx + G_{yy}dy) - G_y(F_{xy}dx + F_{yy}dy),\\
\lambda &= F_y(G_{xx}dx + G_{xy}dy) - G_y(F_{xx}dx + F_{xy}dy),\\
\sigma &= F_x(G_{xx}dx + G_{xy}dy) - G_x(F_{xx}dx + F_{xy}dy),\\
\delta &= F_x(G_{xy}dx + G_{yy}dy) - G_x(F_{xy}dx + F_{yy}dy),
\end{aligned}$$

or

$$\rho = Adx + Bdy, \qquad \lambda = Cdx + Ady, \qquad \sigma = Ddx + Edy, \qquad \delta = Edx + Hdy, \tag{10}$$

where

$$\begin{aligned}
&A = F_yG_{xy} - G_yF_{xy}, \qquad B = F_yG_{yy} - G_yF_{yy}, \qquad C = F_yG_{xx} - G_yF_{xx},\\
&D = F_xG_{xx} - G_xF_{xx}, \qquad E = F_xG_{xy} - G_xF_{xy}, \qquad H = F_xG_{yy} - G_xF_{yy}.
\end{aligned}$$

Thus, from equations (8) and (10),

$$\begin{aligned}
&\alpha = -(Ddx + Edy)/T, \qquad \beta = -(Cdx + Edx + Ady + Hdy)/T,\\
&\gamma = -(Adx + Bdy)/T.
\end{aligned} \tag{11}$$

We now substitute α, β, and γ into equation (7) for dy', divide by dx to convert the differential forms to functions, and rewrite it as:

$$y'' + f_0 + f_1y' + f_2y'^2 + f_3y'^3 = 0, \tag{12}$$

where the f_k are given by

$$f_0 = D/T, \qquad f_1 = (C + 2E)/T, \qquad f_2 = (H + 2A)/T, \qquad f_3 = B/T.$$

We define K and L as

$$K = E/T, \qquad L = A/T, \tag{13}$$

and replace D, C, H, and B in the $1-$forms in equation (11) in favor of the f_k, K, and L, obtaining

$$\alpha = -f_0dx - Kdy, \qquad \beta = (K - f_1)dx + (L - f_2)dy, \qquad \gamma = -Ldx - f_3dy. \tag{14}$$

We also note from equation (6) for dT that now

$$dT/T = (3K - f_1)dx + (f_2 - 3L)dy. \tag{15}$$

We see from the above that it is necessary, for the original assumption of linearizability to hold, that the expression $f(x, y, y')$ in equation (1) be a cubic in y'. Thus the original form of the equation which we have is to be that in equation (12) above, with the f_k known functions of x and y. We see that the $1-$forms α, β, γ, and dT/T are now expressed in terms of these four known functions f_k and two other functions K and L. The first three of these $1-$forms can now be substituted into equations (9) to find conditions on the various functions.

If we do that, the first equation, for $d\alpha$, gives the equation

$$f_{0y} - K_x = -K(K - f_1) + f_0(L - f_2),$$

which is nonlinear in K. The other equations give similar results. However, we can simplify the situation by defining new variables:

$$T = 1/W^3, \qquad E = U/W^4, \qquad A = V/W^4,$$

so that from equation (13)

$$K = U/W, \qquad L = V/W. \tag{16}$$

Equation (15) now becomes

$$3dW/W = (f_1 - 3K)dx + (3L - f_2)dy. \tag{17}$$

We now have this situation. The dW equation (17) gives expressions for W_x and W_y. The $d\alpha$ equation in equation (9) gives, after substitution for W_x, an expression for U_x which is linear in U, V, and W. The $d\gamma$ equation gives an expression for V_y, which is also linear. The $d\beta$ equation gives a linear expression for $V_x - U_y$. The integrability condition on W, $ddW = 0$, gives a linear expression for $V_x + U_y$. The latter two equations can be solved for V_x and U_y. Thus we have expressions for all derivatives of U, V, and W, all of which are linear and homogeneous (no constant terms) in the same variables.

We summarize all these relations in a nice matrix equation

$$dr = Mr, \tag{18}$$

where

$$r = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \qquad \text{and} \qquad M = Pdx + Qdy, \tag{19}$$

where

$$P = (1/3) \begin{bmatrix} -2f_1 & -3f_0 & 3f_{0y} + 3f_0 f_2 \\ 0 & f_1 & 2f_{2x} - f_{1y} - 3f_0 f_3 \\ -3 & 0 & f_1 \end{bmatrix}$$

and

$$Q = (1/3) \begin{bmatrix} -f_2 & 0 & 2f_{1y} - f_{2x} + 3f_0 f_3 \\ 3f_3 & 2f_2 & 3f_{3x} - 3f_1 f_3 \\ 0 & 3 & -f_2 \end{bmatrix}.$$

For integrability, $ddr = 0$, or $0 = dMr - M \wedge dr = dMr - M \wedge Mr$, giving

$$dM = M \wedge M$$

which is not zero since M is a matrix. Substitution for M in terms of P and Q gives the condition

$$Q_x - P_y = [P, Q],$$

the necessary condition on the f_k for linearization to be possible. This matrix condition reduces to two equations:

$$f_{0yy} + f_0(f_{2y} - 2f_{3x}) + f_2 f_{0y} - f_3 f_{0x} + (1/3)(f_{2xx} - 2f_{1xy} + f_1 f_{2x} - 2f_1 f_{1y}) = 0 \tag{20}$$

and

$$f_{3xx} + f_3(2f_{0y} - f_{1x}) + f_0 f_{3y} - f_1 f_{3x} + (1/3)(f_{1yy} - 2f_{2xy} + 2f_2 f_{2x} - f_2 f_{1y}) = 0. \tag{21}$$

To summarize, we note that linearizability requires the original differential equation to be a cubic in y', with the coefficients satisfying equations (20) and (21). These conditions are written out in Lie [1] and in Ibragimov [3], for example.

3 Construction of the linearizing point transformations

In the following, we will need U, V, and W, so we will need to solve equations (18). It is important to note that the most general solution is apparently not necessary; special solutions will suffice. Thus one can make simplifying assumptions in the solution. Once the equations are solved, then we construct K and L from equations (16).

In order to find the $F(x,y)$ and $G(x,y)$ for which we are seeking, we revert to equations (5) and solve for dF_x, dF_y, dG_x, and dG_y. Solution for the first two gives

$$dF_x = (F_y\sigma - F_x\lambda)/T, \qquad dF_y = (F_y\delta - F_x\rho)/T.$$

Solution for the second two, dG_x and dG_y, shows that they satisfy the same equation, so we will write only equations for the derivatives of F. We note that $\delta + \lambda = -T\beta$ and that $\delta - \lambda = dT$, so we can solve these equations for δ and λ. We can also substitute for σ and ρ in terms of α and γ. We get finally

$$dF_x = -F_y\alpha + F_x(\beta + dT/T)/2, \qquad dF_y = F_x\gamma + F_y(-\beta + dT/T)/2.$$

We substitute for α, β, γ, and dW in terms of the expressions obtained above, with the f_k, K, and L. The dW terms disappear and we are left with two equations which we can express in matrix form as follows.

Write

$$R = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} G_x \\ G_y \end{bmatrix}.$$

Now

$$dR = ZR, \qquad dS = ZS, \tag{22}$$

where

$$Z = \begin{bmatrix} (2K - f_1)dx - Ldy & f_0 dx + Kdy \\ -Ldx - f_3 dy & Kdx + (f_2 - 2L)dy \end{bmatrix}.$$

This linear equation set can be solved for R; there will be two independent solutions, which can be taken as R and S. See equation (22). (Integrability is guaranteed by the previous conditions, as can be seen by setting $ddR = 0$.) Finally, one can solve

$$dF = [dx\ dy]R, \qquad dG = [dx\ dy]S \tag{23}$$

for F and G.

We can summarize the procedure.

1. Make sure that the original differential equation is a cubic in y'.
2. Test the coefficients f_k to see whether they satisfy equations (20) and (21). If equations (1) and (2) are satisfied, then the equation is linearizable in principle.
3. Construct the 3×3 matrix M and solve equation (18) (linear!) for the three components of r – a special solution is usually sufficient – and construct K and L.
4. Construct the 2×2 matrix Z and solve equation (22) (linear!) for R or S.
5. Solve equation (23); the two independent solutions may be taken as F and G.

Steps (1) and (2) test for linearizability; steps (3)–(5) perform the construction (in principle).

4 Examples

4.1 The general linear equation

We first consider the equation

$$y'' + a(x)y' + b(x)y + c(x) = 0.$$

We see that $f_2 = f_3 = 0$, $f_1 = a(x)$, and $f_0 = b(x)y + c(x)$. Equations (20) and (21) are satisfied, so this equation can in principle be cast into the form (3). However, when one writes out equation (18), one sees quickly that the resulting linear equations give a second-order equation for U, say, which is as difficult to solve as the original equation. Thus this method is not a magic way to simplify the general linear second-order equation.

4.2 An equation considered by Ibragimov

Ibragimov [3] considered the equation

$$y'' = x^{-1}\left[ay'^3 + by'^2 + \left(1 + b^2/3a\right)y' + b/3a + b^3/27a^2\right],$$

which has $f_3 = -a/x$, $f_2 = -b/x$, $f_1 = -\left(1 + b^2/a\right)/x$, $f_0 = -\left(b/3a + b^3/27a^3\right)/x$, and which satifies the linearizability conditions. Inspection shows that one may take $a = 1$ without loss of generality, and that, by defining new variables U/x, V/x, and W/x^2, one can write an equation like the r equation (18) for which the matrix coefficients are constants, so that it can be solved directly. The details are rather messy, but one eventually gets the linearizing transformation

$$X = F = y + cx, \qquad Y = G = [y + c(x-1)]^2 + x^2,$$

where $c = b/3$. However, this does not save any labor, because the original equation is separable in y' and x and can be integrated quickly!

4.3 A trial equation

We consider the equation

$$y'' + (2/x)y' + \left(18x^2y^3 - 2x/y^2\right)y'^3 = 0, \tag{24}$$

which satisfies the linearizability conditions. We see that $f_0 = f_2 = 0$, $f_1 = 2/x$, and $f_3 = 18x^2y^3 - 2x/y^2$. Thus the matrices P and Q are

$$P = \begin{bmatrix} -4/3x & 0 & 0 \\ 0 & 2/3x & 0 \\ -1 & 0 & 2/3x \end{bmatrix} \qquad \text{and} \qquad Q = \begin{bmatrix} 0 & 0 & 0 \\ 18x^2y^3 - 2x/y^2 & 0 & 2/y^2 \\ 0 & 1 & 0 \end{bmatrix}.$$

From equations (18) and (19), we see that $dU = -(4U/3x)dx$; so we take $U = 0$. Then we have

$$dV = (2V/3x)dx + \left(2W/y^2\right) dy \qquad \text{and} \qquad dW = (2W/3x)dx + V dy,$$

so that $W_x = 2W/3x$. Integrating, we get $W = x^{2/3}a(y)$, for some function $a(y)$. We also see that $V = W_y = x^{2/3}a'(y)$, and further that $a'' = 2a/y^2$. We use the special solution $a(y) = y^2$, yielding finally

$$U = 0, \qquad V = 2x^{2/3}y, \qquad W = x^{2/3}y^2, \qquad \text{so that} \qquad K = 0, \qquad L = 2/y.$$

We can now construct the matrix Z. It is

$$Z = \begin{bmatrix} -2dx/x - 2dy/y & 0 \\ -2dx/y - \left(18x^2\, y^3 - 2x/y^2\right) dy & -4dy/y \end{bmatrix}.$$

Write $R = \begin{bmatrix} b \\ c \end{bmatrix}$. Then from equation (22) we have $db = -2(dx/x + dy/y)b$, which enables immediate integration: $b = k/\left(x^2y^2\right)$, where k is a constant. We also have $c_x = b_y$, which when integrated gives $c = 2k/\left(xy^3\right) + g(y)$.

Finally, we have $c_y = \left(18x^2\, y^3 - 2x/y^2\right) b - 4c/y$, or, after simplification, $g' + 4g/y = -18ky$. Solution gives $g(y) = -3ky^2 + m/y^4$, where m is another constant. Integration of equation (23), $dF = [dx\ dy]R$, now gives two solutions, one proportional to k and the other proportional to m. We take these two solutions as F and G:

$$X = F(x, y) = 1/\left(xy^2\right) + y^3, \qquad Y = G(x, y) = 1/y^3,$$

the linearizing transformation. Construction of d^2Y/dX^2 shows that it is zero provided the original differential equation (24) is satisfied.

Equation (24) was constructed by trial and error in order to provide a useful example of the use of the method. It turns out to have a eight-parameter symmetry group. One can naively try a reduction of order based on a scale transformation together with the usual tricks. Inspection of scale in the equation shows that y has the scale $x^{-1/5}$, so that $y = x^{-1/5}u$ produces the equation

$$x^2u'' + (8/5)xu' - (4/25)u + \left(18u^3 - 2/u^2\right) (xu' - u/5)^3 = 0.$$

We continue by defining $s = \ln x$, $v = du/ds$, and by converting the independent variable to u, with the dependent variable v. We find

$$vdv/du + 3u/5 - 4v/25 + \left(18u^3 - 2/u^2\right) (v - u/5)^3 = 0,$$

a rather nasty Abel equation.

Of course, this naive procedure applied to second order equations in general produces an Abel equation. Application of more sophisticated techniques such as used by Stephani [17] may produce a solution more easily when there are a number of symmetries (which has not been tried here). But the matter does raise the question, is it necessary for an equation to have a certain number of symmetries in order for this method to work well? Ibragimov [3] and Euler [13] note that the answer is yes; a necessary and sufficient condition for linearization by a point transformation is that the equation admit the $sl(3, \mathbb{R})$ Lie point symmetry algebra, or that it admit eight point symmetries. So this is another way to test for linearizability, although the calculation of the symmetries may be lengthy.

4.4 The general Kepler problem

The radial Newtonian central force equation, after substitution for the angular momentum and change of independent variable to θ, can be written

$$\left(\ell/r^2\right)(d/d\theta)\left[\left(\ell/mr^2\right)dr/d\theta\right] - \ell^2/\left(mr^3\right) - f(r) = 0,$$

where $f(r)$ is the force. If $f(r) = -(\ell^2 A/m)r^n$, where A is constant, and we let $r \to y$, $\theta \to x$, we have, where $k = n + 4$,

$$y'' - (2/y)y'^2 - y + Ay^k = 0.$$

Thus $f_3 = 0$, $f_2 = -2/y$, $f_1 = 0$, $f_0 = Ay^k - y$. The linearizability conditions require $k = 1$ or 2 so that $n = -3$ or -2. Why are not more values of n allowed? Because of the restriction of the original assumption of equations (2) and (3).

4.5 Geodesics on a sphere

This equation,

$$y'' = 2y'^2 \cot y + \sin y \cos y,$$

(which also has an eight-parameter symmetry algebra) is treated by Stephani [17, p. 78]. We have

$$f_1 = f_3 = 0, \qquad f_2 = -2\cot y, \qquad f_0 = -\sin y \cos y,$$

and it is easily seen that the linearization conditions are satisfied. A special solution for r gives $V = 0$, $U = \sin x(\sin y)^{2/3}$, $W = \cos x(\sin y)^{2/3}$, so that $K = \tan x$ and $L = 0$. The components of R may be found to be $(b - a\sin x \cot y)(\sec x)^2$ and $a\sec x(\csc y)^2$. We may take the coefficients of a and b to be two independent solutions; then integration for F and G gives $X = F = \tan x$ and $Y = G = \cot y \sec x$, and integration of equation (2) gives

$$\cot y = c\sin x + d\cos x,$$

where c and d are constants, the known solution.

4.6 Example from Stephani

The equation,

$$y'' = (x - y)y'^3,$$

is also treated in Stephani [17] and has an eight-parameter symmetry algebra. One sees easily that the linearization conditions are satisfied. The equation for U is $dU = 0$, so that we may take $U = 0$. Then $dV = -W dy$ and $dW = V dy$, which are satisfied by $V = \sin y$ and $W = -\cos y$, giving $K = 0$ and $L = -\tan y$. Solution for R and S, and then for F and G, gives $F = X = \tan y$ and $G = Y = (x - y)\sec y$. Now $Y = aX + b$, where a and b are constants, gives the solution of the equation essentially as suggested by Stephani:

$$x = y + a\sin y + b\cos y.$$

5 Third-order equation

Some authors have studied the third-order ordinary differential equation [18, 12]. The approach used in the present paper, however, does not readily yield a solution to the third-order problem.

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