Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 2, 623–628

Mirror Symmetry: Algebraic Geometric and Lagrangian Fibrations Aspects

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We survey some algebraic geometric aspects of mirror symmetry and duality in string theory.

1 Introduction

Symmetry principles always played an important role in mathematics and physics. Development of these sciences in direction of string theory enlarged the context of symmetry considerations and included in it the notion of duality. String theory has following ingredients: (i) base space (open or closed string) Σ; (ii) target space M; (iii) fields: $X \to \Sigma \to M$; (iv) action $S = \int \mathcal{L}(X, \varphi)$. where $\mathcal{L}$ is a Lagrangian [1]. Let G be a group such that $G \supset SU(3) \times SU(2) \times U(1)$. Recall that if $\mathcal{L}(G\Phi) = \mathcal{L}(\Phi)$ then $\mathcal{L}$ is G-invariant, or G-symmetry. In string theory [1] one of the beautiful symmetries is the radius symmetry $R \to 1/R$ of circle, known as T-duality [2, 3] and [4] and references there in. Authors of papers [5, 6] conjectured that a similar duality might exist in the context of string propagation on Calabi–Yau (CY) manifolds, where the role of the complex deformation on one manifold gets exchanged with the Kähler deformation on the dual manifold. A pair of manifolds satisfying this symmetry is called *mirror pair*, and this duality is called *mirror symmetry*.

From the point of view of physicists which did the remarkable discovery, mirror symmetry is a type of duality that means that we may take two types of string theory and compactify them in two different ways and achieve “isomorphic” physics [7]. Or in the case of a pair of Calabi–Yau threefolds (X, Y) P. Aspinwall are said [8] that X and Y to be a *mirror pair* if and only if the type IIA string compactified on X is “isomorphic” to the $E_8 \times E_8$ heterotic string compactified on Y. In the case that X is Calabi–Yau threefold Y will be the product of a $K3$ surface and elliptic curve. C. Vafa defines the notion of mirror of a Calabi–Yau manifold with a stable bundle. Lagrangian and special Lagrangian submanifolds appear in this situation. Mathematicians also work hard upon the problems of mirror symmetry, although it is difficult in some cases to attribute to a researcher the identifier “mathematician” or “physicist”. V. Batyrev gives construction of mirror pairs using Gorenstein toric Fano varieties and Calabi–Yau hypersurfaces in these varieties [9]. M. Kontsevich in his talk at the ICM’94 gave a conjecturel interpratation of mirror symmetry as a “shadow” of an equivalence between two triangulated categories associated with A_∞-categories [10]. His conjecture was proved in the case of elliptic curves by A. Polishchuk and E. Zaslow [11]. The aim of the paper is to provide a short and gentle survey of some algebraic aspects of mirror symmetry, duality and special lagrangian fibrations with examples – without proofs, but with (a very restricted) guides to the literature.

2 Preliminaries

We shall use in contrast to [1] some another definition of Calabi–Yau (CY) manifold. The definition based on the theorem of Yau who proved Calabi’s conjecture that a complex Kähler manifold of vanishing first Chern class admits a Ricci-flat metric.

Definition 1. A complex Kähler manifold is called Calabi–Yau (CY) manifold if it has vanishing first Chern class.

Examples of the CY-manifolds include, in particular, elliptic curves E, $K3$-surfaces and their products $E \times K3$.

2.1 Vector bundles

Local chart or a *system of coordinates* on a topological space M is a pair (U, φ) where U is an open set in M and $\varphi : U \to \mathbb{R}^m$ is a homeomorphism from U to an open set $\varphi(U)$ in $\mathbb{R}^m$. An *atlas* Φ of dimension m is a collection of local charts whose domains cover M and such that if $(U, \varphi), (U_1, \varphi_1) \in \Phi$ and $U \cap U_1 \neq 0$ then the map

$$\varphi_1 \circ \varphi^{-1} : \varphi(U \cap U_1) \to \varphi_1(U \cap U_1)$$

is a C^r-diffeomorphism between open sets in $\mathbb{R}^m$.

Fibre space is the object (E, p, B), where p is the continuous surjective (= on) mapping of a topological space E onto a space B and $p^{-1}(b)$ is called the *fibre* above $b \in B$. Both the notation $p : E \to B$ and (E, p, B) are used to denote a *fibration*, a *fibre space*, a *fibre bundle* or a *bundle*.

Vector bundle is fibre space each fibre $p^{-1}(b)$ of which is endowed with the structure of a (finite dimension) vector space V over skew-field K such that the following local triviality condition is satisfied: each point $b \in B$ has an open neighborhood U and a V -isomorphism of fibre bundles $\phi : p^{-1}(U) \to U \times V$ such that $\phi \mid_{p^{-1}(b)}: p^{-1}(b) \to b \times V$ is an isomorphism of vector spaces for each $b \in B$. $\dim V$ is said to be the dimension of the vector bundle.

An *Hermitian bundle* over algebraic variety X consists of a vector bundle over X and a choice of C^∞ Hermitian metric on the vector bundle over complex manifold $X(\mathbb{C})$, which is invariant under antiholomorphic involution of $X(\mathbb{C})$.

The *tangent space* to a differentiable manifold M at point $a \in M$ can be defined as the set of tangency classes of smooth paths in M based at a. It will be denoted by T_aM. Elements of T_aM are called tangent vectors to M at a.

The *tangent bundle* of M, denoted by TM, is the union of the tangent spaces at all the points of M. By well known way TM can be made into a smooth manifold. Recall well known facts about TM :

(i) if M is C^r then TM is C^{r-1};

(ii) if M is C^∞ or C^ω then the same holds for TM;

(iii) if M has dimension n then TM has dimension $2n$;

(iv) there is a natural map $p : TM \to M$ called the *projection* map, taking T_aM to a for each a in M, i.e. p takes all tangent vectors at a to the point a itself. Thus $p^{-1}(a) = T_aM$ (fibre of the bundle over a). The projection p is a smooth map C^{r-1} if M is C^r.

A *vector field* on a smooth manifold M is a map $F : M \to TM$ which satisfies $p \circ F = id_M$, where p is the natural projection $TM \to M$. By its definition a vector field is a *section* of the bundle TM.

2.2 Blow-ups

Blowing up is a well known method of constructing complex manifolds M. There are points on the manifolds that are not divisors on M. Blow up is the construction that transforms points of complex manifolds to divisors. For instance in the case of two dimensional complex manifolds (complex surface) N it consists of replacing a point $p \in N$ by a projective line $\mathbb{CP}(1)$ considered as the set of limit directions at p.

Example 1. Let $\pi : M_2 \to \mathbb{C}^2$ be the blow-up of $\mathbb{C}^2$ at the point $0 \in \mathbb{C}^2$. Then M_2 is a two dimensional complex manifold that defined by two local charts. In coordinates $\mathbb{C}^2 = (z_1, z_2), \mathbb{CP}(1) = [l_0, l_1)$ manifold M_2 is defined in $\mathbb{CP}(1) \times \mathbb{C}^2$ by quadratic equations $z_i l_j = z_j l_i$. Thus M_2 is a line bundle over Riemann sphere $\mathbb{CP}(1)$. $\pi^{-1}(0) = \mathbb{CP}(1)$ is called *the divisor of the blow up (the exceptional divisor).*

Recently a large class of CY orbifolds in weighted projective spaces was suggested. C. Vafa have predicted and S. Roan [14] have computed the Euler number of all the resolved CY hypersurfaces in a weighted projective space $\mathbb{WCP}(4)$.

2.3 Vector bundles over projective algebraic curves

Let X be a projective algebraic curve over algebraically closed field k and g the genus of X. Let $\mathcal{VB}(X)$ be the category of vector bundles over X. Grothendieck showed that for a rational curve every vector bundle is a direct sum of line bundles. Atiyah classified vector bundles over elliptic curves. The main result is

Theorem 1. *Let X be an elliptic curve, A a fixed base point on X. We may regard X as an abelian variety with A as the zero element. Let $\mathcal{E}(r,d)$ denote the the set of equivalence classes of indecomposable vector bundles over X of dimension r and degree d. Then each $\mathcal{E}(r,d)$ may be identified with X in such a way that* $\det : \mathcal{E}(r,d) \to \mathcal{E}(1,d)$ *corresponds to $H : X \to X$, where $H(x) = hx = x + x + \cdots + x$ (h times), and $h = (r,d)$ is the highest common factor of r and d.*

Curve X is called a *configuration* if its normalization is a union of projective lines and all singular points of X are simple nodes [16]. For each configuration X can assign a non-oriented graph $\Delta(X)$, whose vertices are irreducible components of X, edges are its singular and an edge is incident to a vertex if the corresponding component contains the singular point. Drozd and Greuel have proved:

Theorem 2. *1. $\mathcal{VB}(X)$ contains finitely many indecomposable objects up to shift and isomorphism if and only if X is a configuration and the graph $\Delta(X)$ is a simple chain (possibly one point if $X = \mathbb{P}^1$).*

2. $\mathcal{VB}(X)$ is tame, i.e. there exist at most one-parameter families of indecomposable vector bundles over X, if and only if either X is a smooth elliptic curve or it is a configuration and the graph $\Delta(X)$ is a simple cycle (possibly, one loop if X is a rational curve with only one simple node).

3. Otherwise $\mathcal{VB}(X)$ is wild, i.e. for each finitely generated k-algebra Λ there exists a full embedding of the category of finite dimensional Λ-modules into $\mathcal{VB}(X)$.

Let X be an algebraic curve. How to normalize it? There are several methods, algorithms and implementations for this purpose. A new algorithm and implementation is presented in [17].

2.4 Connection

Consider the connection in the context of algebraic geometry. Let S/k be the smooth scheme over field k, U an element of open covering of S, $\mathcal{O}_S$ the structure sheaf on S, $\Gamma(U, \mathcal{O}_S)$ the sections of $\mathcal{O}_S$ on U. Let $\Omega^1_{S/k}$ be the sheaf of germs of 1-dimension differentials, $\mathcal{F}$ be a coherent sheaf. The *connection* on the sheaf $\mathcal{F}$ is the sheaf homomorphism

$$\nabla : \mathcal{F} \to \Omega^1_{S/k} \otimes \mathcal{F},$$

such that, if $f \in \Gamma(U, \mathcal{O}_S)$, $g \in \Gamma(U, \mathcal{F})$ then

$$\nabla(fg) = f\nabla(g) + df \otimes g.$$

There is the dual definition. Let $\mathcal{F}$ be the locally free sheaf, $\Theta^1_{S/k}$ the dual to sheaf $\Omega^1_{S/k}$, $\partial \in \Gamma\left(U, \Theta^1_{S/k}\right)$. The *connection* is the homomorphism

$$\rho : \Theta^1_{S/k} \to \mathrm{End}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}), \qquad \rho(\partial)(fg) = \partial(f)g + f\rho(\partial).$$

3 Moduli spaces in string theory

Mirror symmetry connects with geometrical deformations of complex and Kähler structures on CY-manifolds. So we have to know moduli spaces of complex and Kähler structures on CY-manifolds.

3.1 Moduli spaces

The theory of moduli spaces [12, 13] has, in recent years, become the meeting ground of several different branches of mathematics and physics-algebraic geometry, instantons, differential geometry, string theory and arithmetics. Here we recall some underlieing algebraic structures of the relation. In previous section we have reminded the situation with vector bundles on projective algebraic curves X. On X any first Chern class $c_1 \in H^2(X, \mathbb{Z})$ can be realized as c_1 of vector bundle of prescribed rank (dimension) r. How to classify vector bundles over algebraic varieties of dimension more than 1? This is one of important problems of algebraic geometry and the problem has closed connections with gauge theory in physics and differential geometry. Mamford [12] and others have formulated the problem about the determination of which cohomology classes on a projective variety can be realized as Chern classes of vector bundles? Moduli spaces are appeared in the problem. What is moduli? Classically Riemann claimed that $3g - 3$ (complex) parameters could be for Riemann surface of genus g which would determine its conformal structure (for elliptic curves, when $g = 1$, it is needs one parameter). From algebraic point of view we have the following problem: given some kind of variety, classify the set of all varieties having something in common with the given one (same numerical invariants of some kind, belonging to a common algebraic family). For instance, for an elliptic curve the invariant is the modular invariant of the elliptic curve.

Let $\mathbb{B}$ be a class of objects. Let S be a scheme. A family of objects parametrized by the S is the set of objects

$$X_s : \quad s \in S, \quad X_s \in \mathbb{B}$$

equipped with an additional structure compatible with the structure of the base S. Parameter varieties is a class of moduli spaces. These varieties is a very convenient tool for computer algebra investigation of objects that parametrized by the parameter varieties. We have used the approach for investigation of rational points of hyperelliptic curves over prime finite fields [21].

Example 2. Let ω_1, $\omega_2 \in \mathbb{C}$, $\mathrm{Im}\,(\omega_1/\omega_2) > 0$, $\Lambda = n\omega_1 + m\omega_2$, $n, m \in \mathbb{Z}$ be a lattice. Let H be the upper half plane. Then $H/\Lambda = E$ be the elliptic curve. Let

$$y^2 = x^3 + ax + b = (x - e_1)(x - e_2)(x - e_3), \qquad 4a^3 + 27b^2 \neq 0,$$

be the equation of E. Then the differential of first kind on E is defined by formula

$$\omega = dx/y = dx/\left(x^3 + ax + b\right)^{1/2}.$$

Periods of E:

$$\pi_1 = 2\int_{e_1}^{e_2} \omega, \qquad \pi_2 = 2\int_{e_2}^{e_3} \omega.$$

The space of moduli of elliptic curves over $\mathbb{C}$ is $\mathbb{A}^1(\mathbb{C})$. Its completion is $\mathbb{CP}(1)$.

For $K3$-surfaces the situation is more complicated but in some case is analogous [18].

Theorem 3. *The moduli space of complex structure on market $K3$-surface (including orbifold points) is given by the space of possible periods.*

Some computational aspects of periods and moduli spaces are considered in author's notes [22, 23].

4 Some categorical constructions

Let (X, ω, Ω) be a complex manifold (real dimension $=2n$) with

$$\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n \cdot \Omega \wedge \overline{\Omega}.$$

It is said that a n-dimensional submanifold $L \subset X$ is *special Lagrangian (s-lag)* $\Leftrightarrow$

$$\mathrm{Re}\,(\Omega|_L) = \mathrm{Vol}|_L \;\Leftrightarrow\; \omega|_L = 0, \qquad \mathrm{Im}\,(\Omega|_L) = 0.$$

Example 3. Let X be an elliptic curve E. Then $\omega = c(i/2)dz \wedge d\overline{z}$, $\Omega = cdz$. S-lag $L \subset E$ are straight lines with slope determined by $\arg c$.

Every compact symplectic manifold Y, ω with vanishing first Chern class, one can associate a A_∞-category whose objects are essentially the Lagrangian submanifolds of Y, and whose morphisms are determined by the intersections of pairs of submanifolds. This category is called Fukaya's category and is denoted by $\mathcal{F}(Y)$ [10]. Let (X, Y) be a mirror pair. Let M be any element of the mirror pair. The bounded derived category $D^b(M)$ of coherent sheaves on M is obtained from the category of bounded complexes of coherent sheaves on M [19]. In the case of elliptic curves A. Poleshchuk and E. Zaslov have proved [11]:

Theorem 4. *The categories $D^b(E_q)$ and $\mathcal{F}^0(\overline{E}^q)$ are equivalent.*

Recently A. Kapustin and D. Orlov have suggested that Kontsevich's conjecture must be modified: coherent sheaves must be replaced with modules over Azumaya algebras, and the Fukaya category must be "twisted" by closed 2-form [20].

Acknowledgements

I would like to thank the organizers of the SymmNMPh'2001 and SAGP'99 (Mirror Symmetry in String Theory, CIRM, Luminy) for providing a very pleasant environment during the conferences.

[1] Greene B., String theory on Calabi–Yau manifolds, hep-th/9702155.

[2] Dixon L., Lecture at the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology, Trieste, 1987.

[3] Lerche W., Vafa C. and Warner N., Chiral rings in $N = 2$ superconformal theories, *Nucl. Phys. B*, 1989, V.324, 421–430.

[4] Giveon A., Porrati M. and Rabinovici E., Target space duality in string theory, *Phys. Rept.*, 1994, N 244, 77–202.

[5] Candelas P., de la Ossa P., Green P., and Parkes L., A pair of Calabi–Yau manifolds as an exactly solvable superconformal theory, *Nucl. Phys. B*, 1991, V.359, 21–74.

[6] Greene B., Morrison D., and Plesser M., Mirror manifolds in higher dimension, *Commun. Math. Phys.*, 1995, N.173, 559–598.

[7] Vafa C., Geometric physics, in Proceedinds of ICM'98, Berlin, *Documenta Math. J. DMV*, 1998, V.1, 430–439.

[8] Aspinwall P., String theory and duality, in Proceedinds of ICM'98, Berlin, *Documenta Math. J. DMV*, 1998, V.2, 229–238.

[9] Batyrev V., Mirror symmetry and toric geometry, in Proceedinds of ICM'98, Berlin, *Documenta Math. J. DMV*, 1998, V.2, 239–248.

[10] Kontsevich M., Homological algebra of mirror symmetry, in Proceedinds of ICM'94, Zurich, Birkhauser, 1995, 120–139.

[11] Polishuk A. and Zaslow E., Categorical mirror symmetry: the elliptic curves, *Adv. Theor. Math. Phys.*, 1998, N 2, 443–470.

[12] Mamford D., Towards an enumirative geometry of the moduli space of curves, in Arithmetic and Geometry, Vol.II, Progress in Math., 1983.

[13] Harris J. and Morrison J., Moduli of curves, Berlin – New York, Springer, 1998.

[14] Roan S., Algebraic geometry and physics, math-ph/0011038.

[15] Atiyah M., Vector bundles over elliptic curves, *Proc. London Math. Soc.*, 1957, V.7, N 27, 414–452.

[16] Drozd Ju. and Greuel G., On vector bundles over projective curves, in Proc. of Int. Conf. “Representation Theory and Computer Algebra” (18–23 March, 1997, Kyiv), Kyiv, Insitute of Mathematics, 1997, 11.

[17] Decker W., de Jong T., Greuel G. and Pfister G., The normalization: A new algorithm, implementation and comparisons. in Proceedings of the Euroconference “Computational Methods for Representations of Groups and Algebras” (1–5, April, 1997, Essen), Basel, Birkhäuser, 1999, 177–185.

[18] Barth W., Peters C., and van der Ven., Compact complex surfaces, New York, Springer, 1984.

[19] Gelfand S.I. and Manin Yu.I., Methods of homological algebra, Moscow, Nauka, 1988.

[20] Kapustin A. and Orlov D., Remarks on A-branes, mirror symmetry, and the Fukaya category, hep-th/0109098.

[21] Glazunov N.M., On moduli spaces, equidistribution, bounds and rational points of algebraic curves, *Ukr. Math. J.*, 2001, N 9, 1174–1183.

[22] Glazunov N.M., Some algebraic geometric aspects of Calabi–Yau manifolds and their zeta functions, in Abstracts of Int. Workshop “Computer Algebra and its Applications to Physics (CAAP-2001)”, Dubna, 2001, 18.

[23] Glazunov N.M., Computation of moduli, periods and modular symbols, IMACS-ACA'98, Session “Algebra and Geometry”, Prague, 1998; http://math.unm.edu/ACA/1998/sessions/geom/glazunov/index.html.