Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 2, 540–547

Quantum Algebras, Particle Phenomenology, and (Quasi)Supersymmetry

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Quantum algebras $U_q(\mathrm{su}_n)$ used as the algebras of flavour symmetry (usually described by $SU(n)$) to study static properties of hadrons lead to intriguing results. In this contribution we focus on the peculiar properties manifested by different q-deformed structures (e.g., the braided line, the quantum algebras $U_q(\mathrm{su}_2)$ and $U_q(\mathrm{su}_n)$, $n \geq 3$) in the special limit of $q = -1$. Similarities (complete or partial) with supersymmetry that emerge in this special limit are discussed.

1 Introduction

Our goal is to pay special attention to the exotic situation that arises if, within the application of quantum algebras $U_q(\mathrm{su}_n)$ [1, 2] to phenomenological description (see [3, 4] and refs. therein) of basic static properties of hadrons – vector mesons as well as baryons, one restricts itself to the peculiar case $q = -1$ of the deformation parameter. In the paper, we first briefly mention the two more or less realistic appearances of supersymmetry (SUSY) algebras applied directly in the sector of hadron mass spectrum. Note that the first appearance of SUSY in the context of hadron physics goes back to Miyazawa's paper [5]. It employs a kind of superalgebra which is connected with internal symmetry and extends the usual $SU(3)$ scheme by means of baryon number changing currents. In that paper, the author has succeeded to derive, based on a superalgebra, the mass sum rules other than the celebrated Gell-Mann–Okubo (GMO) one, that is, $m_N + m_\Xi = \frac{3}{2}m_\Lambda + \frac{1}{2}m_\Sigma$. On the contrary, the spectrum generating (or dynamical) superalgebra used in [6] incorporated a superization of space-time symmetry and gave a possibility to analyse the towers of excited states, for each ground state baryon (e.g., nucleon) or vector meson (e.g., ρ-meson). We discuss these two examples in Section 2. Then, Sections 3 and 4 are devoted to the very instructive examples of q-deformed structure which, if one sends $q \to -1$, show either exact SUSY (the case of braided line whose relation to SUSY is considered in Section 3), or the features only reminiscent of supersymmetry, see Section 4. In the 5th section we deal with the peculiar case of $q = -1$ concerning the quantum algebras $U_q(\mathrm{su}_n)$ which appear in the context of their use as the algebras describing flavor symmetries of hadrons and enabling to derive new, very precise mass relations. In this scheme, the restriction to the limit $q = -1$ is physically motivated.

2 Dynamical supersymmetry and hadron mass spectrum

In [5] the two copies of superalgebra, namely,

$$\begin{aligned}
&[F_i, F_j] = \mathrm{i} f_{ijk} F_k, \qquad [F_i, G_j] = \mathrm{i} f_{ijk} G_k, \qquad \{G_i, G_j\} = d_{ijk} F_k, \\
&[\bar{F}_i, \bar{F}_j] = \mathrm{i} f_{ijk} \bar{F}_k, \qquad [\bar{F}_i, \bar{G}_j] = \mathrm{i} f_{ijk} \bar{G}_k, \qquad \{\bar{G}_i, \bar{G}_j\} = -d_{ijk} \bar{F}_k,
\end{aligned} \tag{1}$$

have been introduced. For their realization, the conventional 3×3 Hermitian matrices λ_i ($i = 0, 1, 2, 3, 8$ for the F_i, $\bar{F}_i$, and $i = 4, 5, 6, 7$ for the G_i, $\bar{G}_i$) have been utilized. By means of symmetry breaking terms (C^{3b}_{3b}, C^{a3}_{a3} and C^{33}_{33} in the notation of [5]) which provide mass splitting between quarks and diquarks (i.e., SUSY breaking), as well as splitting between isomultilplets (breaking of $SU(3)$ to $SU(2)$), instead of the standard GMO mass relation the formulas

$$m_N + m_\Xi = m_\Lambda + m_\Sigma, \qquad m_{Y^*_0} = m_\Sigma, \qquad 2m_{K^*} = m_\rho + m_\phi, \qquad m_\rho = m_\omega \tag{2}$$

for baryons and for vector mesons have been obtained.

A completely different scheme for treating hadron mass spectrum developed in [6] employs a particular *dynamical superalgebra* Osp(1|4) connected with space-time symmetries. The dynamical superalgebra with generators $S_{\mu\nu}$, Γ_μ, Q_α respects the chain

$$\mathrm{Osp}(1|4)_{S_{\mu\nu},\Gamma_\mu,Q_\alpha} \supset \mathrm{SO}(3,2)_{S_{\mu\nu},\Gamma_\mu} \supset \mathrm{SO}(3,1)_{S_{\mu\nu}},$$

where for the subalgebras $\mathrm{SO}(3,1)_{S_{\mu\nu}}$ and $\mathrm{SO}(3,2)_{S_{\mu\nu},\Gamma_\mu}$ the generators $S_{\mu\nu}$ and Γ_μ obey

$$[S_{\mu\nu}, S_{\rho\sigma}] = -i(\eta_{\mu\rho}S_{\nu\sigma} + \eta_{\nu\sigma}S_{\mu\rho} - \eta_{\nu\rho}S_{\mu\sigma} - \eta_{\mu\sigma}S_{\nu\rho}), \tag{3}$$

$$[S_{\mu\nu}, \Gamma_\rho] = -i(\eta_{\mu\rho}\Gamma_\nu - \eta_{\nu\rho}\Gamma_\mu), \qquad [\Gamma_\mu, \Gamma_\nu] = -iS_{\mu\nu}. \tag{4}$$

The relations involving anticommuting charges Q_α and $\bar{Q}_\beta = -(Q^T C)_\beta$, namely

$$[S_{\mu\nu}, Q_\alpha] = -\frac{1}{2}\left(\sigma^s_{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta, \qquad [\Gamma_\mu, Q_\alpha] = -\frac{1}{2}\left(\gamma_\mu\right)_\alpha^{\ \beta} Q_\beta,$$
$$\{Q_\alpha, \bar{Q}_\beta\} = -\frac{1}{2}\left(\sigma^{\mu\nu}\right)_{\alpha\beta} S_{\mu\nu} + \left(\gamma^\mu\right)_{\alpha\beta}\Gamma_\mu,$$

along with (3), (4), complete the symmetry algebra to the superalgebra $\mathrm{Osp}(1|4)_{S_{\mu\nu},\Gamma_\mu,Q_\alpha}$.

To construct the Hamiltonian, supercharges should be incorporated (like in supersymmetric quantum mechanics), through the term $\frac{1}{2n}\sum\limits_{\alpha=1}^{n}\{Q_\alpha, Q^\dagger_\alpha\}$. The resulting Hamiltonian

$$H = v\left(P_\mu P^\mu - \frac{1}{\alpha'}\frac{1}{4}\sum_{\beta=1}^{4}\{Q_\beta, Q^\dagger_\beta\} - \tilde{m}^2_0\right)$$

is to be completed by Casimirs of subalgebras in the chain $\mathrm{Osp}(1|4) \supset \mathrm{SO}(3,2)_{S_{\mu\nu},\Gamma_\mu} \supset \mathrm{SO}(3)_{S_{ij}} \times \mathrm{SO}(2)_{\Gamma_0}$. In its final form, the Hamiltonian reads

$$H = v\left(P_\mu P^\mu - \frac{1}{\alpha'}\hat{P}_\mu\Gamma^\mu - \lambda^2\hat{W} + \beta\hat{C}_{\mathrm{SO}(3,2)} - \tilde{m}^2_0\right) \tag{5}$$

and, correspondingly, hadron mass spectrum is described by the formula [6]

$$m^2 = -\frac{1}{\alpha'}\mu + \lambda^2 j(j+1) + \beta\left(2 - 2s^2\right) + \tilde{m}^2_0. \tag{6}$$

In this expression, $1/\alpha'$ (related to the slope of Regge trajectory), λ^2, and β are empirical system parameters; μ resp. $j(j+1)$ are eigenvalues of $\hat{P}_\mu\Gamma^\mu$ resp. $\hat{W}$; s labels $\mathrm{SO}(3,2)$ representations, and $\tilde{m}_0$ is the background mass.

Comparison of the mass formula (6) with experimental data, using the particular representation $D\left(\frac{3}{2},\frac{1}{2}\right) \oplus D(2,1)$ of the dynamical superalgebra, shows that the series (tower) of excited states over the lowest lying 1^- vector mesons ρ or ω and the $\frac{1}{2}^+$ nucleon's tower (its resonances) fit the data very well if one sets: $\frac{1}{\alpha'}(\text{meson}) \sim \frac{1}{\alpha'}(\text{nucleon})$ and $\lambda^2(\text{meson}) \sim \lambda^2(\text{nucleon})$. It is this fact that was interpreted in [6] as a kind of empirical evidence for supersymmetry in the hadron mass spectra. This observation may be considered as an extension of the well-known success of dynamical supersymmetries in nuclear physics [7] to the level of hadrons.

3 q-deformed oscillator at $q \to -1$ and supersymmetry

In [8] it was shown that the q-deformed calculus on the braided line [9] (tightly connected with q-deformed oscillator), in the nontrivial particular case of $q = -1$ exhibits supersymmetric properties. In this section we discuss some details of this correspondence, following [8].

The braided (or q-deformed) line is defined [9] in terms of a single non-commuting variable θ which obeys a Hopf algebra structure operating with coproduct,

$$\Delta\theta = \theta \otimes 1 + 1 \otimes \theta, \tag{7}$$

$$(1 \otimes \theta)(\theta \otimes 1) = q\theta \otimes \theta, \qquad (\theta \otimes 1)(1 \otimes \theta) = \theta \otimes \theta. \tag{8}$$

as well as a counit and antipode. Note that it is the first relation in (8) that determines the nontrivial (for $q \neq 1$) braiding.

With $[X, Y]_z \equiv XY - zYX$, denoting $\theta = 1 \otimes \theta$ and $\delta\theta = \epsilon = \theta \otimes 1$ as in Ref. [9], yields

$$[\epsilon, \theta]_{q^{-1}} = 0 \qquad \text{and} \qquad \Delta\theta = \epsilon + \theta.$$

Here the latter equality corresponds to (7); it encodes the action upon θ of the left translation by ϵ, $L_\epsilon\theta :\ \theta \mapsto \epsilon + \theta$. As seen, ϵ and θ anticommute when $q = -1$.

To construct a differential calculus on the braided line, one introduces a left derivation operator with respect to θ, obeying $[\epsilon\mathcal{D}_L, \theta] = \epsilon$, so that

$$[\mathcal{D}_L, \theta] = 1, \qquad \frac{d}{d\theta}\theta = 1. \tag{9}$$

Likewise, one can introduce right shifts $R_\eta\theta :\ \theta \mapsto \theta + \eta$ by odd parameter η so that $[\theta, \eta]_{q^{-1}} = [\eta, \theta]_q = 0$ (again, θ and η anticommute if $q = -1$). The right derivative operator satisfies $[\theta, \mathcal{D}_R] = 1$ and also the relation

$$\mathcal{D}_R = -q^{-(1+N)}\mathcal{D}_L \tag{10}$$

involving the number operator N defined according to

$$[N, \theta] = \theta, \qquad [N, \mathcal{D}_L] = -\mathcal{D}_L. \tag{11}$$

The differential calculus defined by (9)–(11) at generic q is called q-calculus.

With the identification $\theta = a^\dagger$, $\mathcal{D}_L = q^{N/2}a$, the q-calculus is related to the q-deformed harmonic oscillator [10]

$$aa^\dagger - q^{\mp 1/2}a^\dagger a = q^{\pm N/2}. \tag{12}$$

The entity $q^{1/2}$ and its power $\left(q^{1/2}\right)^N$ in (12) are of importance since, from (12), by exploiting Hermitian conjugacy one comes to the formulas $aa^\dagger = [N+1]_{q^{1/2}}$ and $a^\dagger a = [N]_{q^{1/2}}$ valid for the q-deformed oscillator [10] of Biedenharn and Macfarlane. Here $[A]_z \equiv \left(z^A - z^{-A}\right) / \left(z - z^{-1}\right)$.

Let $\lfloor A \rfloor_q \equiv (1-q^A)/(1-q)$. A function of θ given by the expansion $f(\theta) = \sum\limits_{m=0}^{\infty} C_m\theta^m/\lfloor m \rfloor_q!$ admits the derivative $\frac{d}{d\theta}f(\theta) = \sum\limits_{m=0}^{\infty} C_m\theta^m/\lfloor m-1 \rfloor_q!$ implying that

$$\left[\mathcal{D}_L, \frac{\theta^m}{\lfloor m \rfloor_q!}\right]_{q^m} = \frac{\theta^{m-1}}{\lfloor m-1 \rfloor_q!}.$$

The difficulties appearing in the limit $q \to -1$ already at $m = 2$ (since $\lfloor 2 \rfloor_q = 0$ in this limit) are tamed by setting $q = -1 + iy$ and letting $y \to 0$. Then, the definition

$$t := \lim_{q \to -1} \left(i\theta^2/\lfloor 2 \rfloor_q!\right) \tag{13}$$

implying that, with $\theta^2 = 0$ imposed, the limit of the ratio in (13) should be finite and nonzero, imports the additional variable t as a necessary ingredient of the braided line if $q \to -1$. As shown in [8], in this limit the terms of the form $\theta^{2r+p}/\lfloor 2r+p \rfloor_q!$ also can be handled by means of t. Due to this, any function $f(\theta)$ on the braided line (generic q), reduces in the limit $q \to -1$ to a 'superfield' given by the function $f(t,\theta)$.

It can be shown that $[\mathcal{D}_L^2, t] = i$ and, with the definition

$$\{\mathcal{D}_L, \mathcal{D}_L\} = 2i\partial_t \qquad \text{or} \qquad \partial_t = -i\mathcal{D}_L^2,$$

the relation $[\partial_t, t] = 1$ is valid. The operator $\mathcal{D}_L$ then becomes the supercharge, $\mathcal{D}_L \equiv Q$, of one-dimensional supersymmetry, and one comes to the relations:

$$Q = \partial_\theta + i\theta\partial_t, \qquad \{Q, Q\} = 2i\partial_t.$$

Likewise, the operator $D = \mathcal{D}_R = (-1)^N \mathcal{D}_L$ becomes the (super)covariant derivative so that

$$D = \partial_\theta - i\theta\partial_t, \qquad \{D, D\} = -2i\partial_t, \qquad \text{and} \qquad \{Q, D\} = 0.$$

Another interesting result derived in [8] is the coproduct for t with unusual θ-dependent term:

$$\Delta t = t \otimes 1 + 1 \otimes t + i\theta \otimes \theta.$$

Thus, proper treatment of braided line in the peculiar limit $q \to -1$ shows that, in this limit, an additional variable t related to θ^2 (see (13)), as well as to higher powers, must arise. As a result, the braided line at $q \to -1$ is made up of the two variables θ and t which span the one-dimensional superspace, SUSY being the translational invariance along this line.

4 Example of Zachos, based on the $q = -1$ limit of $U_q(\mathrm{su}_2)$

Quantum algebra $U_q(\mathrm{su}_2)$ [1, 2] is generated by the elements I_+, I_-, I_0, obeying the relations

$$\begin{aligned} &[I_0, I_\pm] = \pm I_\pm, \qquad [I_+, I_-] = [2J_0]_q \equiv \left(q^{2J_0} - q^{-2J_0}\right) / \left(q - q^{-1}\right), \\ &\Delta(J_0) = J_0 \otimes 1 + 1 \otimes I_0, \qquad \Delta(J_\pm) = J_\pm \otimes q^{-J_0} + q^{+J_0} \otimes J_\pm \end{aligned} \tag{14}$$

and the relations that involve antipode and counit (which will not be used here).

As shown in [11], this quantum algebra exhibits an intriguing features at the level of its representations when the deformation parameter $q = -1$. Let us consider this example.

Using coproduct, one can form composites of two spin $\frac{1}{2}$ doublets according to $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$:

$$\begin{array}{lll} \text{singlet} & \longleftrightarrow & \alpha = |q^{1/2} \uparrow\downarrow - q^{-1/2} \downarrow\uparrow\rangle, \\ \text{triplet} & \longleftrightarrow & \left\{ \begin{array}{l} \beta = |\uparrow\uparrow\rangle, \\ \Delta(J_-)\beta = \frac{1}{\sqrt{2}} |q^{1/2} \uparrow\downarrow + q^{-1/2} \downarrow\uparrow\rangle, \\ (\Delta(J_-))^2 \beta = |\downarrow\downarrow\rangle. \end{array} \right. \end{array}$$

For $q = 1$, the singlet state is antisymmetric whereas each of the triplet states is symmetric. Now let $q = -1$. In this case the multiplets turn into

$$\begin{array}{ll} \alpha = |i \uparrow\downarrow - \frac{1}{i} \downarrow\uparrow\rangle & \text{(symmetric)}, \\ \beta = |\uparrow\uparrow\rangle & \text{(symmetric)}, \\ \Delta(J_-)\beta = \frac{1}{\sqrt{2}} |i \uparrow\downarrow + \frac{1}{i} \downarrow\uparrow\rangle & \text{(antisymmetric)}, \\ (\Delta(J_-))^2 \beta = |\downarrow\downarrow\rangle & \text{(symmetric)}. \end{array} \tag{15}$$

It is seen from (15) that the coproduct operation $\Delta(J_-)$ changes the symmetry of wave function. That is, rasing and lowering operators in the coproduct act as statistics-altering operators. Although the constituents of the states haven't been converted to fermions, this alteration of the symmetry of wave function *is reminiscent of* SUSY. It is instructive to compare this structure with $N=2$ supersymmetric quantum mechanics, stressing both similarities and peculiar features.

Consider (graded) direct product of two copies of SUSY QM algebras:

$$\begin{aligned} &SS^\dagger + S^\dagger S = 1, \qquad ss^\dagger + s^\dagger s = 1, \qquad S^\dagger S^\dagger = s^\dagger s^\dagger = SS = ss = 0, \\ &sS + Ss = 0, \qquad s^\dagger S^\dagger + S^\dagger s^\dagger = 0, \qquad sS^\dagger + S^\dagger s = 0, \qquad s^\dagger S + Ss^\dagger = 0. \end{aligned} \tag{16}$$

This graded Lie algebra can be obtained, using appropriate Wigner–Inonü contraction, from the simple Lie superalgebra $SU(2|1)$ (realizable in terms of Gell-Mann $SU(3)$ λ-matrices so that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$ constitute even generators whereas $\{\lambda_4, \lambda_5, \lambda_6, \lambda_7\}$ constitute odd generators).

One can realize the algebra (16) on two boson states $|B\rangle$, $|b\rangle$, and two fermion states $|F\rangle$, $|f\rangle$, as: $S\ |B\rangle = |F\rangle$, $s|b\rangle = |f\rangle$, $S^\dagger|F\rangle = |B\rangle$, $s^\dagger|f\rangle = |b\rangle$. The (nullifying) rest of actions reads: $S\ |F\rangle = S|b\rangle = s|B\rangle = s^\dagger|F\rangle = s^\dagger|b\rangle = S^\dagger|f\rangle = S^\dagger|B\rangle = s|f\rangle = 0$. With their use,

$$s|Bb + bB\rangle = |Bf + fB\rangle, \qquad Ss|Bb + bB\rangle = |Ff - fF\rangle. \tag{17}$$

Thus, $\Delta(J_-)$ in (15) switches the symmetry of wave function like the even (bosonic) operator $Ss = -sS$, see (17), but only the latter is nilpotent due to nilpotency of S, s. The other important difference consists in the structure and dimensionality of multiplets. Namely, for $q=-1$ these remain the same as in the classical case of $su(2)$ Lie algebra. On the other hand, for graded Lie algebra the representations are of different dimensions (compare, e.g., $SU(2|1)$ and $SU(3)$). Hence, the conclusion: this $q=-1$ case implies a kind of *quasi-supersymmetry.*

5 GMO formula and $U_q(su_n)$ at $q=-1$

One can either utilize representation-theoretic aspects of the quantum algebra $U_q(su_n)$ or, alternatively, construct the mass operator using q-tensor operators. In the latter case [12], main ingredients of the Hopf algebra structure of $U_q(su_n)$ (comultiplication Δ and antipode S) play the role. The Δ and S are defined [1, 2] on the $U_q(su_n)$ generators $E_i^\pm$ and H_i as

$$\begin{aligned} &S\left(E_i^\pm\right) = -q^{H_i/2} E_i^\pm, \qquad S(H_i) = -H_i, \qquad S\left(q^{H_i/2}\right) = q^{-H_i/2}, \qquad S(1) = 1, \\ &\Delta\left(E_i^\pm\right) = E_i^\pm \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_i^\pm, \qquad \Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i q^{-H_i/2}. \end{aligned} \tag{18}$$

The adjoint action of $U_q(su_n)$ defined [2] as $\mathrm{ad}_A B = \sum A_{(1)} B S(A_{(2)})$ with $A, B \in U_q(su_n)$ and $A_{(1)}$, $A_{(2)}$ determined from $\Delta(A) = \sum A_{(1)} \otimes A_{(2)}$, with the account of (18) reads:

$$\begin{aligned} &\mathrm{ad}_{H_i} B = H_i B 1 + 1 B S(H_i) = H_i B - B H_i, \\ &\mathrm{ad}_{E_i^\pm} B = E_i^\pm B q^{-H_i/2} - q^{-H_i/2} B q^{H_i/2} E_i^\pm q^{-H_i/2}. \end{aligned}$$

The q-tensor operators [13] transforming under the adjoint action of $U_q(su_3)$ as $\mathbf{3}$ and $\mathbf{3}^*$, consist of the triples (V_1, V_2, V_3) and $(V_{\bar 1}, V_{\bar 2}, V_{\bar 3})$, respectively. Let $[X, Y]_q \equiv XY - qYX$. It can be shown that the particular triple of elements from $U_q(su_4)$

$$\begin{aligned} &V_1 = [E_1^+, [E_2^+, E_3^+]_q]_q\ q^{-H_1/3 - H_2/6}, \\ &V_2 = [E_2^+, E_3^+]_q\ q^{H_1/6 - H_2/6}, \qquad V_3 = E_3^+ q^{H_1/6 + H_2/3} \end{aligned} \tag{19}$$

transform as **3** under $U_q(su_3)$, V_1 corresponds to the highest weight vector, the pair (V_1, V_2) is $U_q(su_2)$ (iso)doublet and V_3 its singlet. Likewise one constructs from elements of $U_q(su_4)$ the triple $(V_{\bar 1}, V_{\bar 2}, V_{\bar 3})$ that transforms as $\mathbf{3}^*$ under adjoint action of $U_q(su_3)$, where $V_{\bar 3}$ corresponds to the highest weight vector, the pair $(V_{\bar 1}, V_{\bar 2})$ is isodoublet and $V_{\bar 3}$ is $U_q(su_2)$ singlet.

The mass operator $\hat M = \hat M_0 + \hat M_8$ involves $\hat M_0$, as $U_q(su_3)$ scalar, and the term $\hat M_8$ transforming as the $I = 0, Y = 0$ component of tensor operator of **8**-irrep of $U_q(su_3)$. The irrep **8** occurs twice in the decomposition $\mathbf{8}\otimes\mathbf{8} = \mathbf{1}\oplus\mathbf{8}^{(1)}\oplus\mathbf{8}^{(2)}\oplus\mathbf{10}^*\oplus\mathbf{10}\oplus\mathbf{27}$. Then, usage of Wigner–Eckart theorem for $U_q(su_n)$ quantum algebras [13] applied to q-tensor operators transforming as irrep **8** of $U_q(su_3)$, turns the mass operator into $\hat M = \hat M_0 + \hat M_8 = M_0\mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)}$. Here **1** is the identity operator, $V_8^{(1)}$ and $V_8^{(2)}$ are two fixed tensor operators with non-proportional matrix elements, each transforming as the $I = 0, Y = 0$ component of irrep **8** of $U_q(su_3)$; M_0, α and β are some constants depending on details (dynamics) of the model.

If $|B_i\rangle$ is a basis vector of representation **8** space which corresponds to some $(1/2)^+$ baryon, then the mass of this baryon is calculated as

$$M_{B_i} = \langle B_i|\hat M|B_i\rangle = \langle B_i|\left(M_0\mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)}\right)|B_i\rangle. \tag{20}$$

The decompositions $\mathbf{3}\otimes\mathbf{3}^* = \mathbf{1}\oplus\mathbf{8}$, $\mathbf{3}^*\otimes\mathbf{3} = \mathbf{1}\oplus\mathbf{8}$ imply that the operators $V_3V_{\bar 3}$ and $V_{\bar 3}V_3$ formed from V_3 in (19) and $V_{\bar 3}$ are just the two isosinglets $V_8^{(1)}$, $V_8^{(2)}$ needed in (20). Hence, the mass operator in (20) can be rewritten (redefining M_0, α, β) in the equivalent form

$$\hat M = M_0\mathbf{1} + \alpha V_3V_{\bar 3} + \beta V_{\bar 3}V_3 = \hat M = M_0\mathbf{1} + \alpha E_3^+E_3^- q^Y + \beta E_3^-E_3^+ q^Y, \tag{21}$$

where the hypercharge $Y = (H_1 + 2H_2)/3$ has been introduced.

To calculate matrix elements (20) using (21) we embed the octet **8** in a particular irrep of $U_q(su_4)$; embedding it, e.g., in **15** (adjoint) irrep of $U_q(su_4)$, we get the octet baryon masses

$$M_N = M_0 + \beta q, \qquad M_\Sigma = M_0, \qquad M_\Lambda = M_0 + [2]_q[3]_q^{-1}(\alpha+\beta), \qquad M_\Xi = M_0 + \alpha q^{-1} \tag{22}$$

(obviously, the expressions for M_N, M_Ξ are not invariant under $q \to q^{-1}$). Excluding M_0, α and β from (22) results in the following q-analogue of GMO formula for octet baryons:

$$[3]_q M_\Lambda + M_\Sigma = [2]_q\left(q^{-1}M_N + qM_\Xi\right). \tag{23}$$

Using empirical masses, the deformation parameter q is fixed by fitting: for each of the $q_{1,2} = \pm 1.035$, $q_{3,4} = \pm 0.903\sqrt{-1}$, the q-deformed mass relation (23) holds within experimental uncertainty (although for q_3, q_4 the constants α and β in (22) must be pure imaginary).

The right hand side of equation (23) is invariant under $q \to q^{-1}$ only if $q = q^{-1}$, that is, if $q = \pm 1$. Behind the 'classical' GMO mass formula which obviously follows from (23) at $q = 1$ and corresponds to the nondeformed unitary symmetries $SU(4) \supset SU(3) \supset SU(2)$, there is also an unusual 'hidden symmetry' reflecting the singular $q = -1$ case of $U_q(su_4) \supset U_q(su_3) \supset U_q(su_2)$ algebras, undefined in this case. The relevant objects, however, exist as operator algebras [12]. Let us describe them in the part corresponding to $n = 2$ and $n = 3$.

At generic q, $q \neq -1$, the algebra $U_q(su_2)$ is generated by the elements E^+, E^- and H, which satisfy the relations

$$[H, E^\pm] = \pm 2E^\pm, \qquad [E^+, E^-] = [H]_q. \tag{24}$$

In the limit $q \to 1$ it reduces to the nondeformed su_2. We take the representation spaces of the latter in order to construct operator algebras for the case $q = -1$. To each su_2 representation space given by j (which takes integral or half-integral nonnegative values) with basis elements

$|jm\rangle$, $m = -j, -j+1, \ldots, j$, there corresponds an operator algebra generated by the operators defined according to the formulas

$$H|jm\rangle = 2m|jm\rangle, \qquad E^+|jm\rangle = \alpha_{j,m}|jm+1\rangle, \qquad E^-|jm\rangle = \alpha_{j,m-1}|jm-1\rangle, \tag{25}$$

where

$$\alpha_{j,m} = \begin{cases} \sqrt{-(j-m)(j+m+1)}, & j \quad \text{is an integer,} \\ \sqrt{(j-m)(j+m+1)}, & j \quad \text{is a half-integer.} \end{cases}$$

So defined operators E^+, E^- and H *on the basis elements* $|jm\rangle$ satisfy the relations (compare with (24)), one of which *depends on whether j is an integer or a half-integer*:

$$[H, E^\pm] = \pm 2E^\pm, \qquad [E^+, E^-] = \begin{cases} -H, & j \quad \text{is an integer;} \\ H, & j \quad \text{is a half-integer.} \end{cases} \tag{26}$$

To treat the (singular) case $q = -1$ of $U_q(su_3)$ it is more convenient to deal with $U_q(u_3)$. We take a representation space V_χ, labelled by $\{m_{13}, m_{23}, m_{33}\} \equiv \chi$, of the nondeformed u_3 and the Gel'fand–Tsetlin basis with the basis elements $|\chi; m_{12}, m_{22}; m_{11}\rangle$ in each V_χ. Define the operators E_1^+, E_1^-, H_1, E_2^+, E_2^-, H_2 that form the operator algebra of the χ-type by their action according to the formulas (let us denote $\sigma_{1,3} \equiv m_{11} + m_{13} + m_{23} + m_{33}$):

$$\begin{aligned} H_2|\chi; m_{12}, m_{22}; m_{11}\rangle &= (2m_{12} + 2m_{22} - m_{13} - m_{23} - m_{33} - m_{11})|\chi; m_{12}, m_{22}; m_{11}\rangle, \\ E_2^+|\chi; m_{12}, m_{22}; m_{11}\rangle &= a_{\chi,m_{11}}(m_{12}, m_{22})\ |\chi; m_{12}+1, m_{22}; m_{11}\rangle \\ &\quad + b_{\chi,m_{11}}(m_{12}, m_{22})\ |\chi; m_{12}, m_{22}+1; m_{11}\rangle, \\ E_2^-|\chi; m_{12}, m_{22}; m_{11}\rangle &= a_{\chi,m_{11}}(m_{12}-1, m_{22})\ |\chi; m_{12}-1, m_{22}; m_{11}\rangle r \\ &\quad + b_{\chi,m_{11}}(m_{12}, m_{22}-1)\ |\chi; m_{12}, m_{22}-1; m_{11}\rangle, \end{aligned}$$

where

$$a_{\chi,m_{11}}(m_{12}, m_{22}) = \left((-1)^{\sigma_{1,3}} \frac{(m_{13} - m_{12})(m_{23} - m_{12} - 1)(m_{33} - m_{12} - 2)(m_{11} - m_{12} - 1)}{(m_{22} - m_{12} - 1)(m_{22} - m_{12} - 2)} \right)^{1/2},$$

$$b_{\chi,m_{11}}(m_{12}, m_{22}) = \left((-1)^{\sigma_{1,3}} \frac{(m_{13} - m_{22} + 1)(m_{23} - m_{22})(m_{33} - m_{22} - 1)(m_{11} - m_{22})}{(m_{12} - m_{22} + 1)(m_{12} - m_{22})} \right)^{1/2}.$$

Action formulas for the operators $E_1^\pm$ and H_1 are completely analogous to formulas (25) above (with account of $m_{11} - m_{22} = 2j$, $2m_{11} - m_{12} - m_{22} = 2m$).

The presented action formulas for the operators that form the operator algebra of the χ-type show that their matrix elements are, to some extent, similar to the 'classical' matrix elements (i.e. to the matrix elements of the irrep χ operators for $su(n)$). However, there is an essential distinction: now we observe the important phase factors (namely, $(-1)^{m_{11}+m_{13}+m_{23}+m_{33}}$ under the square root in $a_{\chi,m_{11}}$ and $b_{\chi,m_{11}}$) which *depend on χ and a specified basis element.* No such basis-element dependent factors exist in the $su(n)$ case.

Let us illustrate such treatment with the particular example of operator algebra appearing in the singular $q = -1$ case of $U_q(su_3)$ and corresponding to the octet representation of su_3. We give here explicitly only those action formulas for $E_1^\pm$ and $E_2^\pm$ in which matrix elements differ from their corresponding 'classical' counterparts:

$$\begin{aligned} &E_1^-|\Sigma^+\rangle = \sqrt{-2}|\Sigma^0\rangle, \qquad E_1^-|\Sigma^0\rangle = \sqrt{-2}|\Sigma^-\rangle, \qquad E_1^+|\Sigma^-\rangle = \sqrt{-2}|\Sigma^0\rangle, \\ &E_1^+|\Sigma^0\rangle = \sqrt{-2}|\Sigma^+\rangle, \qquad E_2^-|n\rangle = \frac{1}{\sqrt{-2}}|\Sigma^0\rangle + \sqrt{-3/2}|\Lambda\rangle, \qquad E_2^-|\Lambda\rangle = \sqrt{-3/2}|\Xi^0\rangle, \end{aligned}$$

$$E_2^-|\Sigma^0\rangle = \frac{1}{\sqrt{-2}}|\Xi^0\rangle, \qquad E_2^+|\Xi^0\rangle = \frac{1}{\sqrt{-2}}|\Sigma^0\rangle + \sqrt{-3/2}|\Lambda\rangle, \qquad E_2^+|\Lambda\rangle = \sqrt{-3/2}|n\rangle,$$
$$E_2^+|\Sigma^0\rangle = \frac{1}{\sqrt{-2}}|n\rangle.$$

To complete this operator algebra, we must add the rest of action formulas for $E_1^\pm$ and $E_2^\pm$ (i.e., action on those basis elements) which coincide with the 'classical' ones, as well as the action formulas for H_1, H_2 (these latter also coincide with 'classical' formulas).

Likewise, for $U_q(su_3)$ at $q = -1$ an operator algebra corresponding to any other irrep of su_3 can be given. The treatment is obviously extendible to $U_{q=-1}(su_n)$, $n > 3$.

Let us also remark that SUSY-based mass relation $m_\rho = m_\omega$, see (2), is obtainable from a q-deformed structure. Indeed, it follows from the q-analog of vector meson mass relation,

$$m_{\omega_8} + (2[2]_q/[3]_q - 1)m_\rho = (2[2]_q/[3]_q)m_{K^*}$$

(which was derived [14] using $U_q(su_n)$ quantum algebras), if one fixes q as 4th root of unity: $q = \sqrt{-1}$ (then, $[2]_q = 0$). The intriguing interplay between SUSY and the special cases $q = -1$ and $q = \sqrt{-1}$ of the q-algebras $U_q(su_n)$ deserves further detailed study.

Acknowledgement

The research described in this paper was made possible in part by Award No. UP1-2115 of the U.S. Civilian Research and Development Foundation for Independent States of the Former Soviet Union (CRDF).

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