Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 1, 128–135

Symmetries, Singularities and Integrability in Nonlinear Mathematical Physics and Cosmology

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An overview is given of the interplay between symmetries, singularities and integrability and their uses in nonlinear problems arising in Mathematical Physics and Cosmology. A particularly important aspect is the role of nonlocal symmetries in deciding about integrability of complex nonlinear problems which do not apparently admit solutions in closed form. The need for a new approach to the evolution of symmetries themselves is also discussed.

1 Concepts of integrability

In all the areas of Mathematical Modelling which give rise to differential equations the modelling process includes the solution of those differential equations, be they (systems of) ordinary differential equations or partial differential equations. If this be possible in some sense, the system of differential equations is said to be integrable. (Note that we exclude numerical integration since this requires merely the existence of a continuous solution and that property can even be found in chaotic/turbulent systems.) A critical question is the meaning of "in some sense". There are four possible ways to prescribe integrability. They are

(i) the ability to display a nonlocal functional equation involving the dependent and independent variables; this need not be explicit and, should the equation be implicit, the inversion by means of the Implicit Function Theorem need be no more than local,

(ii) the existence of a number of functionally independent first integrals/invariants equal to the order of the system in general and half that for a Lagrangian system as a consequence of Liouville's Theorem [1],

(iii) the existence of a sufficient number of Lie symmetries to reduce the differential equation (or system; unless otherwise obviously the singular implies the plural) to an algebraic equation and

(iv) the possession of the Painlevé Property.

These concepts are not entirely equivalent. In particular (*iv*) requires that the solution be analytic or possess no more than algebraic branch points in the complex plane (planes for more than one independent variable) and this is not demanded by (*i*), (*ii*) and (*iii*) although, of course, the idea that a solution must be analytic to be considered as a solution has been with us since the days of Poincaré. Even (*i*) and (*ii*) are not equivalent since it is not always possible to eliminate nonlocally the derivatives from the functionally independent first integrals/invariants. Case (*iii*) differs from (*i*) and (*ii*) since the final algebraic equation is in terms of the invariants of the symmetries used in the reduction of order and the reversal of the process – on the assumption that a nonlocal solution of the algebraic equation exists – requires a series of quadratures which one may not be able to perform in closed form. In the case of Lagrangian systems the celebrated theorem of Noether [2] allows the identification of (*ii*) and (*iii*). The precise nature of the relationship between (*iii*) and (*iv*) has yet to be revealed although some recent work points to a subtler relation than previously expected [3–7].

2 Evolution of symmetry

When Lie introduced his ideas of symmetry based upon the geometry of infinitesimal transformations [8], the symmetries were naturally in the variables of the extended configuration space. With his introduction of contact transformations [9] the variables of the transformation became those of the extended phase space. For Lie both point and contact symmetries were seen in the context of the geometry of a space of finite dimensions. The adoption of generalised symmetries by Noether removed this constraint, particularly in the case of partial differential equations. (The order of the equation for an ordinary differential equation provides an effective bound in that case.) The inclusion of nonlocal symmetries was necessitated by the observation of the so-called "hidden symmetries" in which "regular symmetries" seemed to appear from nowhere on the lowering or raising of the order of an equation. To take a trivial example, in the change of order

$$Y''' = 0 \quad \Leftrightarrow \quad y'' = 0; \quad x = X, \quad y = Y'$$

the point symmetries

$$\gamma_1 = x^2\partial_x + xy\partial_y, \qquad \Gamma_2 = X^2\partial_X + 2XY\partial_Y$$

of the latter and former equations respectively come from the nonlocal symmetries

$$\Gamma_1 = X^2\partial_X + 3\left(XY - \int Y\mathrm{d}X\right)\partial_Y, \qquad \gamma_2 = x^2\partial_x + 2\left(\int y\mathrm{d}x\right)\partial_y$$

of the former and latter respectively.

When one accepts the generality of form implied by a nonlocal symmetry, there is as little need for the imputed esoterica of 'hidden' as there is to distinguish between geometrical and dynamical symmetries in Mechanics.

A feature of the Lie symmetries of a differential equation is that they constitute an algebra, a representation of a group, and the algebra is used to place a given differential equation in an equivalence class. As a trivial example all scalar second order ordinary differential equations have eight point symmetries with the algebra $sl(3,\mathbb{R})$ and so belong to the equivalence class of $y'' = 0$. In the case of $y'' = 0$ not all of those eight symmetries are required to specify it completely. There is, as it were, an oversupply of symmetry for the specification just as there is for the integrability, for, if we require the equation

$$y'' = f(x, y, y')$$

to possess the three symmetries, just three of the eight point symmetries constituting the elements of $sl(3,\mathbb{R})$,

$$\gamma_1 = \partial_x, \qquad \gamma_2 = \partial_y, \qquad \gamma_3 = x\partial_y,$$

the right hand side is constrained to be zero. Any scalar second order ordinary differential equation is completely specified by three symmetries [11].

When Krause introduced the concept of a complete symmetry group [12], the vehicle for his exposition was the Kepler problem with the equation of motion

$$\ddot{\boldsymbol{r}} + \frac{\mu\hat{\boldsymbol{r}}}{r^2} = 0, \qquad r^2 = x^2 + y^2 + z^2 \tag{1}$$

which possesses the five Lie point symmetries

$$\Gamma_1 = y\partial_z - z\partial_y, \qquad \Gamma_2 = z\partial_x - x\partial_z, \qquad \Gamma_3 = x\partial_y - y\partial_x,$$
$$\Gamma_4 = \partial_t, \qquad \Gamma_5 = 3t\partial_t + 2r\partial_r$$

with the algebra $A_2 \oplus so(3)$. These five symmetries are insufficient to specify completely (1) and Krause found it necessary to find the three nonlocal symmetries

$$\Gamma_6 = \left(\int x\,\mathrm{d}t\right)\partial_t + xr\partial_r, \qquad \Gamma_7 = \left(\int y\,\mathrm{d}t\right)\partial_t + yr\partial_r, \qquad \Gamma_8 = \left(\int z\,\mathrm{d}t\right)\partial_t + zr\partial_r,$$

a type of generalised conformal symmetry, to complete the task. Subsequently Nucci [13] obtained these nonlocal symmetries by standard local methods. Nucci and Leach [14] added an additional six nonlocal symmetries obtained by means of a reduction for the Kepler Problem based on the Ermanno–Bernoulli components of the Laplace–Runge–Lenz vector and showed that similar results were obtained for other systems possessing a conserved vector analogous to the Laplace–Runge–Lenz vector.

In the gradual evolution of the concept of a symmetry – a process of over a century – there have been both gains and losses. The gains have included an increased variety of systems that can be integrated using symmetries and a greater understanding of the rôle played by symmetry in integrability. For example the generalisation of the Hénon–Heiles problem [15] with Hamiltonian

$$H = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + Ax^2 + By^2\right) + D^2y - \frac{1}{3}Cy^3$$

is known to be integrable in the cases that $C = -2D$, $C = -6D$ and $C = -D$. Clearly the existence of one first integral, the Hamiltonian, is due to the symmetry ∂_t. The existence of a second first integral is due to the existence of another point symmetry in the first two cases. For the third the responsible symmetry is the nonlocal symmetry [16]

$$\Gamma = y\partial_t + \dot{y}\left(2x - F\right)\partial_x + y\partial_y,$$

where, in the coefficient function of ∂_x, F is the nonlocal term given by

$$F = \int \frac{\dot{x}\dot{y} + xy(1+2x)}{\dot{y}^2}\,\mathrm{d}t.$$

In the computation using the Lie method of the first integral

$$I = \dot{x}\dot{y} + xy + \frac{1}{3}x^3 + xy^2$$

that coefficient is not used.

An even more dramatic example is found in the trivially integrated

$$yy'' - y'^2 = f'y^{n+2} + nfy'y^{n+1}, \tag{2}$$

which was advanced as an integrable equation devoid of symmetry [17, 18]. By means of the simple, albeit nonlocal, transformation

$$X = x, \qquad Y = -\int nfy^n\,\mathrm{d}x + \log\left[-\int nfy^n\,\mathrm{d}x\right] - \log f$$

(2) becomes

$$\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} = 0$$

which possesses eight Lie point symmetries with the algebra $sl(3, \mathbb{R})$. These Lie point symmetries find expression as nonlocal symmetries for (2).

There are two areas of loss. In the first instance the ease of calculation of Lie point symmetries and its algorithmic implementation in symbolic manipulation codes is lost when one seeks nonlocal symmetries and somewhat diminished in the cases of contact and generalised symmetries. This very practical problem is likely to maintain the popularity of point, contact and generalised symmetries for many years to come. At a more elevated mathematical level is the problem of deciding between those symmetries which are useful and those which are useless. How does one decide if a nonlocal symmetry is useful or not? Exponential nonlocal symmetries are fine for determining invariants [19] but not for reduction of order since the reverse procedure is not a matter of quadratures [20]. We have instanced above examples in which one would not credit the nonlocal symmetry as having more than curiosity value and yet integrability results. The resolution of this question is one of the more difficult theoretical problems in the study of symmetry. For the nonce one's choice of the type of symmetry to use is more than likely to be based upon utilitarianism than generality [21].

3 Putting symmetry to work

We illustrate the uses of symmetry in resolving some classes of problems which arise in Mathematical Physics and Cosmology.

There exist hierarchies of integrable partial differential equations which have attracted considerable attention over the last forty years. One of these of more recent interest is the hierarchy of evolution equations

$$u_t = R^m[u]\left(u^{-2}u_x\right)_x, \tag{3}$$

where the recursion operator

$$R[u] = D_x^2 u^{-1} D_x^{-1}$$

generates the hierarchy. This hierarchy has been shown to be linearisable, to possess an infinite number of symmetries and autohodograph transformations [22, 23]. The class (3) possesses four Lie point symmetries [24], *videlicet*

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = \partial_x, \qquad \Gamma_3 = (m+2)t\partial_t + u\partial_u, \qquad \Gamma_4 = -x\partial_x + u\partial_u$$

and these may be used to reduce the $1+1$ evolution equation to a nonlinear ordinary differential equation. A suitable choice for the reduction is [25]

$$\Gamma = \frac{1}{m+2}\Gamma_3 + (m+1)\Gamma_4 = t\partial_t - \frac{m+1}{m+2}x\partial_x + u\partial_u$$

(Γ_1 and Γ_2 could be included to allow for translation in t and x, but here we are illustrating a point and not essaying an exhaustive study.) and, since the reduced equation inherits a scaling symmetry, a further transformation based on that symmetry leads to the autonomous equation

$$\mathrm{e}^{-T}R[X]^m\mathrm{e}^T\left[-\left(\frac{1}{X}\right)' + \frac{1}{X}\right]' + \frac{m+1}{m+2}\dot{X} - \frac{1}{m+2}X = 0,$$

where

$$R[X] = -\left(\mathrm{e}^T\right)^2 X^{-1}\mathrm{e}^{-T}D_T^{-1}\mathrm{e}^{-T}$$

and the prime represents differentiation with respect to the new independent variable T, in which one notes that there is a preservation of the recursion property. The overall transformation is

$$T = -\log xt^{\frac{m+1}{m+2}}, \qquad X = uxt^{\frac{2m+3}{m+2}}.$$

We conclude an example taken directly from Cosmology [26]. The general Lagrangian leading to the full Bianchi-scalarfield dynamics (that is Einstein equations for an homogeneous but anisotropic spacetime with a scalarfield matter source with a self-interacting potential $V(\phi)$) has the form

$$\mathcal{L} = \mathrm{e}^{3\lambda}\left[R^* + 6\dot{\lambda}^2 - \frac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{\phi}^2 + 2V(\phi)\right], \tag{4}$$

where R^* is the Ricci scalar playing the role of a potential term, β_1 and β_2 are suitable variables describing the anisotropy and derivatives are taken with respect to proper time t. The Euler–Lagrange equations for (4) are

$$\begin{aligned}
&\ddot{\lambda} + \frac{3}{2}\dot{\lambda}^2 + \frac{3}{8}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) + \frac{1}{4}\dot{\phi}^2 - \frac{1}{12}\mathrm{e}^{-3\lambda}\left(\mathrm{e}^{3\lambda}R^*\right)_\lambda - \frac{1}{2}V(\phi) = 0,\\
&\ddot{\beta}_1 + 3\dot{\beta}_1\dot{\lambda} + \frac{1}{3}\frac{\partial R^*}{\partial \beta_1} = 0,\\
&\ddot{\beta}_2 + 3\dot{\beta}_2\dot{\lambda} + \frac{1}{3}\frac{\partial R^*}{\partial \beta_2 0} = 0,\\
&\ddot{\phi} + 3\dot{\phi}\dot{\lambda} + V' = 0.
\end{aligned}$$

For homogeneous Bianchi Class A models the Ricci scalar R^* has the explicit form

$$\begin{aligned}
R^* = {} & -\frac{1}{2}\mathrm{e}^{-2\lambda}\left[N_1^2\mathrm{e}^{4\beta_1} + \mathrm{e}^{-2\beta_1}\left(N_2\mathrm{e}^{\sqrt{3}\beta_2} - N_3\mathrm{e}^{-\sqrt{3}\beta_2}\right)^2\right.\\
& \left. -2N_1\mathrm{e}^{2\beta_1}\left(N_2\mathrm{e}^{\sqrt{3}\beta_2} + N_3\mathrm{e}^{-\sqrt{3}\beta_2}\right)\right] + \frac{1}{2}N_1N_2N_3(1 + N_1N_2N_3)
\end{aligned}$$

and for Class B universes

$$R^* = 2a^2\mathrm{e}^{-2\lambda}\left(3 - \frac{N_2N_3}{a^2}\right)\mathrm{e}^{\beta}$$

with

$$\beta = \frac{2}{3a^2 - N_2N_3}\left(N_2N_3\beta_1 + \sqrt{-3a^2N_2N_2}\beta_2\right),$$

where a, N_1, N_2 and N_3 are the usual classification constants. For the symmetry analysis it is convenient to make the substitutions

$$u = \mathrm{e}^{\lambda}, \qquad v = \mathrm{e}^{\beta_1}, \qquad w = \mathrm{e}^{\sqrt{3}\beta_2}.$$

We illustrate the results for the simplest Bianchi Type I models and for the open Bianchi Type V family in the case of a constant scalar field potential, *i.e.* $V(\phi) = C$.

In the case of Bianchi Type I with a constant potential the Noether point symmetries are

$$\begin{aligned}
&\partial_t, \qquad v\partial_v, \qquad w\partial_w, \qquad \partial_\phi, \qquad v\log w\partial_v - 3w\log v\partial_w,\\
&v\phi\partial_v - \frac{3}{2}\log v\partial_\phi, \qquad w\phi\partial_w - \frac{1}{2}\log w\partial_\phi.
\end{aligned}$$

In addition there are the three Lie point symmetries

$$u\partial_u, \qquad \mathrm{e}^{\sqrt{3}Ct}\left\{\partial_t + u\partial_u\right\}, \qquad \mathrm{e}^{\sqrt{3}Ct}\left\{\partial_t - u\partial_u\right\}.$$

We find the first integrals/invariants (listed against the corresponding symmetry)

$$\begin{aligned}
&v\partial_v, && I_1 = u^3\dot{v}/v,\\
&w\partial_w, && I_2 = u^3\dot{w}/w,\\
&\partial_\phi, && I_3 = u^3\dot{\phi},\\
&v\log w\partial_v - 3w\log v\partial_w, && I_4 = \frac{\dot{u}^2}{u\dot{\phi}} - \frac{u^3\dot{v}}{4v\dot{\phi}} - \frac{\dot{\phi}}{6},\\
& && I_5 = t - \alpha\,\mathrm{arcsinh}\,\frac{u^3+M}{\beta},
\end{aligned}$$

where

$$\alpha = \frac{2}{3\sqrt{K}}, \qquad \beta = \left[\frac{2I_3^2}{3K^2}\left(K - 6I_4^2\right) - \frac{C}{2K}\right]^{1/2}, \qquad M = \frac{2I_3I_4}{K}, \qquad K = \frac{I_1^2+I_2^2}{I_3^2}.$$

By inverting the invariant I_5 we obtain $u(t)$ and hence $v(t)$, $w(t)$ and $\phi(t)$ from the quadrature of the first three integrals. Thus we have an explicit solution for this model.

For the Bianchi Type V in the case of a constant potential we obtain the Noether point symmetries

$$\partial_t, \qquad v\partial_v, \qquad w\partial_w, \qquad \partial_\phi$$

and the additional Lie point symmetries

$$v\log w\partial_v - 3w\log v\partial_w, \qquad v\phi\partial_v - \frac{3}{2}\log v\partial_\phi, \qquad w\phi\partial_w - \frac{1}{2}\log w\partial_\phi.$$

We obtain the integrals

$$\begin{aligned}
&\partial_t, && I_1 = u^3\dot{v}/v,\\
& && I_2 = u^3\dot{w}/w,\\
& && I_3 = u^3\dot{\phi},\\
&v\partial_v, && I_4 = \frac{1}{2}(\log u)^3\frac{\dot{u}^2}{u} - f(u),
\end{aligned}$$

where

$$\begin{aligned}
f(u) = {}& \frac{1}{16u^2}\left[4(\log u)^3 + 6(\log u)^2 + 6\log u + 3\right]\\
&+ \frac{C}{8}(\log u)^4\frac{1}{48u^6}\left(3I_1^2 + I_2^2 + 2I_3^2\right)\left[(\log u)^3 + \frac{1}{2}(\log u)^2 + \frac{1}{6}\log u + \frac{1}{36}\right].
\end{aligned}$$

In contrast to Type I one is left with the quadrature of I_4 and inversion of the result to obtain $u(t)$. This is not possible in closed form and so we have a system which is integrable but for which an explicit global solution is not available.

4 Discussion

In this paper we provided an overview of the interplay between three fundamental notions of dynamics, namely, symmetry (local and nonlocal) singularities and integrability. There are many questions that remain open in this field some of which come about from considerations arising when one tries to apply the results obtained from the calculations of symmetries to decide about the integrability of the given family of nonlinear systems. For example we know that the cosmological solutions discussed above evolve to other solutions in the limit of large times. This evolution is usually one from a complex (for instance anisotropic) early time state to a simpler late time, isotropic one. It is also true that in such cases an originally nonintegrable system evolves asymptotically to an integrable one. This fact raises an interesting point regarding symmetries and integrability: If symmetry is indeed needed as a fundamental ingredient of the integrability properties of an arbitrary nonlinear system, this has to somehow show in its long term evolution. How do the calculated symmetries of a system evolve as the system changes in time? Almost all work on symmetry and integrability to date has been concerned, in some sense, only with the "statics" of the problem. We believe that only a theory of the dynamical evolution of the symmetries themselves as a given system evolves in time will be needed to provide the means to understand and explain why particular systems of differential equations have the complicated symmetry properties they appear to have. As such a theory is completely lacking at present, examples that show in a clear way the road to proceed will be most welcome.

Acknowledgements

PGLL thanks the Research Lab of Geometry, Dynamical Systems and Cosmology (GEODYSYC) for kind hospitality when this work was initiated and acknowledges the continued support of the National Research Foundation of the Republic of South Africa and the University of Natal.

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