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Smoothness Properties of Green's–Samoilenko Operator-Function the Invariant Torus of an Exponentially Dichotomous Bilinear Matrix Differential System

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In this paper the smoothness properties of Green's operator-function an exponentially dichotomous bilinear matrix system and the smoothness properties the invariant torus of nonhomogeneus matrix system of equations have been considered. It hHave been proved that if some conditions, concerning the properties of coefficient of the system hold this operator-function has smoothness index which depends on both the smoothness of matrix coefficients of the system and their spectral properties.

We consider the system of equations

$$\frac{d\phi}{dt} = a\left(\phi\right), \qquad \frac{dX}{dt} = A\left(\phi\right)X - XB\left(\phi\right) + F\left(\phi\right), \tag{1}$$

where $a^T(\phi) = (a_1(\phi), a_2(\phi), \ldots, a_m(\phi))$, $\phi^T = (\phi_1, \phi_2, \ldots, \phi_m)$, $\phi_i \in [0, 2\pi)$, $i = \overline{1, m}$, are vectors, $A(\phi) = A_{n_1\times n_1}$, $B(\phi) = B_{n_2\times n_2}$, $F(\phi) = F_{n_1\times n_2}$, $X = X_{n_1\times n_2}$ are matrix functions defined and continuous with respect to $\phi \in T_m$, where $T_m = T_1 \times T_1 \times \cdots \times T_1$ is m-dimensional torus, $\phi_i \in T_1 = [0, 2\pi)$, $i = \overline{0, m}$. We shall call the system (1) a matrix bilinear non-homogeneous system of equations defined on a direct product of m-dimensional torus T_m and the space of matrices $M_{n_1\times n_2}$, under assumption that spectral sets of matrices $A(\phi)$ and $B(\phi)$ satisfy the condition $\sigma(A(\phi)) \bigcap \sigma(B(\phi)) = \varnothing$, and the system (1) is exponentially dichotomous. We define the norm of matrix in the space $M_{n_1\times n_2}$ as Frobenius or trace-norm $\|X\|^2 = \mathrm{tr}(X^*X)$. The Green's operator-function for the system of equations (1) defined by relation

$$[G_t(\tau, \phi)]F(\phi_\tau(\phi)) = \begin{cases} [\Omega_0^t(\phi)][P_1(\phi)][\Omega_\tau^0(\phi)]F(\phi_\tau(\phi)), & t \geq \tau, \\ -[\Omega_0^t(\phi)][P_2(\phi)][\Omega_\tau^0(\phi)]F(\phi_\tau(\phi)), & t < \tau. \end{cases} \tag{2}$$

where $[\Omega{}^t_\tau(\phi)]Z = \underset{A}{\Omega}{}^t_\tau(\phi)Z\underset{B}{\Omega}{}^\tau_t(\phi)$, $[P_k(\phi)]Z = \sum P_{i_k}(\phi)ZQ_{j_k}(\phi)$, (k=1,2), $\underset{A}{\Omega}{}^t_\tau(\phi)$, $\underset{B}{\Omega}{}^t_\tau(\phi)$ are matricents of matrix differential equation associated with matrix A and B accordingly. $P_i(\phi)$, $Q_j(\phi)$ are projection operators to proper subspace in Euclidean space E_{n_1} and E_{n_2}. $[P_1]$, $[P_2]$ are projection operators in the space of matrices $M_{n_1\times n_2}$, $[P_1(\phi)] + [P_2(\phi)] = [I]$, $[I]$ is the identity operator in the space $M_{n_1\times n_2}$, $[I]Z = I_{n_1}ZI_{n_2} = Z$, $\eta_{i_k,j_k}(A(\phi), B(\phi)) = \lambda_{i_k}(A(\phi)) - \mu_{j_k}(B(\phi))$ is an eigenvalue of operator $\Phi(\phi)X = A(\phi)X - XB(\phi)$, $k = 1$ when $\eta_{i_k,j_k}(A(\phi), B(\phi)) < 0$ and $k = 2$ when η_{i_k,j_k} $(A(\phi), B(\phi)) > 0$, $\lambda_{i_k}(A(\phi))$ and $\mu_{j_k}(B(\phi))$ are eigenvalues of matrices $A(\phi)$ and $B(\phi)$ accordingly. The solution of the second homogeneous matrix equation (1) has the form [1] $X_t(\phi, X) = \underset{A}{\Omega}{}^t_\tau(\phi)X\underset{B}{\Omega}{}^\tau_t(\phi) = [\Omega{}^t_\tau(\phi)]X$. The operator $[\Omega{}^t_\tau(\phi)]$ in the space $M_{n_1\times n_2}$ has the group property $[\Omega{}^t_\tau(\phi_\theta(\phi))] = [\Omega{}^{t+\theta}_{\tau+\theta}(\phi)]$ that follows from the properties of matricents $\underset{A}{\Omega}{}^t_\tau(\phi)$, $\underset{B}{\Omega}{}^t_\tau(\phi)$ [2].

We call $[G_t(\tau,\phi)]$ a Green's operator-function for system of equations (1) in the case when integral

$$\int_{-\infty}^{\infty} \|[G_0(\tau,\phi)]\|\, d\tau \le K < \infty$$

is uniformly bounded with respect to ϕ. We give the simplest properties of the Green's operator-function. It follows from its definition that $[G_0(\tau,\phi)] \in C(T_m)$ for $\forall\ \tau$ and $[G_0(-0,\phi)] - [G_0(+0,\phi)] = [P_1(\phi)] + [P_2(\phi)] = [I]$. Suppose that matricents $\underset{A}{\Omega}{}^t_\tau(\phi)$, $\underset{B}{\Omega}{}^t_\tau(\phi)$ satisfy inequalities

$$\|\underset{A}{\Omega}{}^t_\tau(\phi)\| \le K_1 \exp(-\gamma_1(t-\tau)), \qquad \|\underset{B}{\Omega}{}^t_\tau(\phi)\| \le K_2 \exp(-\gamma_2(\tau-t)), \qquad t > \tau, \tag{3}$$

for all $\phi \in T_m$, and some positive K_i, γ_i, $(i = 1, 2)$ independent of ϕ. From (3) the estimate follows $(t > \tau)$

$$\|[G_t(\tau,\phi)]F(\phi_\tau)\| \le \|\underset{A}{\Omega}{}^t_\tau(\phi)\|\, \|F(\phi_\tau)\|\, \|\underset{B}{\Omega}{}^\tau_t(\phi)\| \le Ke^{-(\gamma_1-\gamma_2)(t-\tau)}\, \|F(\phi_\tau(\phi)\|\,. \tag{4}$$

We suppose that homogeneous system of equations (1) is exponentially dichotomous, then a Green's operator-function satisfies the estimate [3, 4]

$$\|[G_t(\tau,\phi)]\| \le Ke^{-(\gamma_1-\gamma_2)|t-\tau|}, \qquad t,\tau \in \mathbb{R}, \quad \phi \in T_m, \tag{5}$$

where $K > 0$, $\gamma = \gamma_1 - \gamma_2 > 0$ are positiv constants independent of ϕ.

From estimate (5) the existence of invariant matrix torus of the system (1) follows, which is given by the relation

$$U(\phi) = \int_{-\infty}^{0} [\Omega^t_\tau(\phi)][P_1(\phi_\tau(\phi))]F_\tau(\phi)d\tau - \int_0^\infty [\Omega^t_\tau(\phi)][P_2(\phi_\tau(\phi))]F_\tau(\phi)d\tau. \tag{6}$$

The smoothness of invariant torus (6) of the system (1) depends essentially on the properties of the Green's function $[G_0(\tau,\phi)]$ and the solution $\phi_t(\phi)$ of the first equation of the system, which defines a trajectory flow for system (1) on the torus $U(\phi)$ [2]. We need to have the estimate of derivative $\partial\phi_t(\phi)/\partial\phi_j$ which is equal j-th column Jacobi matrix for vector-function $\phi_t(\phi)$, which is satisfying the system of equations $d\theta/dt = a'(\phi)\theta$, where $a'(\phi) = D\phi_t(\phi)/D\phi$ is the matrix of partial derivative of the function $\phi_t(\phi)$ or Jacobi matrix. We denote $\underset{a}{\Omega}{}^t_0(\phi)$ matricient of this system, it is characterized the stability of solutions of a nonperturbed system on a torus [2]. For obtaining of estimate of derivatives of operator-function $[G_t(\tau,\phi)]$ in ϕ_i we use the estimate of derivatives $\partial^s\phi_t(\phi)/\partial^s\phi_i = D^s_{\phi_i}\phi_t(\phi)$ which was obtained in [2].

$$\|D^s_{\phi_i}\phi_t(\phi)\| \le Ke^{(s\alpha+\varepsilon)|t|}, \qquad t \in \mathbb{R}, \quad \phi \in T_m, \tag{7}$$

Taking $s = 1$ we obtain the estimate $\|\underset{a}{\Omega}{}^t_0(\phi)\| \le Ke^{(\alpha+\varepsilon)|t|}$. The estimate of derivatives of operator-function $[G_t(\tau,\phi)]$ is essentially defined by smoothness properties of invariant torus of the nonhomogeneous system of equations and depends on smoothness of coefficients of the system (1) $a(\phi)$, $A(\phi)$, $B(\phi)$ and spectral properties of matrices $A(\phi)$, $B(\phi)$.

Theorem 1. *Assume that for some integer positive number $l \ge 0$ the following conditions holds: $A(\phi) \in C^l_{\mathrm{Lip}}(T_m)$, $B(\phi) \in C^l_{\mathrm{Lip}}(T_m)$, $a(\phi) \in C^l_{\mathrm{Lip}}(T_m)$ and $\tilde\gamma = \gamma_1 - \gamma_2 - \varepsilon \ge l\alpha$, where $\alpha > 0$, $\varepsilon > 0$ is an arbitrary small positive number. Then*

$$\|D^s_\phi[G_0(\tau,\phi)]\| \le Ke^{-(\gamma_1-\gamma_2-\varepsilon-s\alpha)|\tau|}, \tag{8}$$

where $0 \le s \le l$, $K = K(\varepsilon)$ is a positive constant independent of $\phi \in T_m$.

Proof. Because $e^{\varepsilon|\tau|} > 1$ for $\tau \neq 0$, then $e^{-\gamma|\tau|} < e^{-\gamma|\tau|+\varepsilon|\tau|} = e^{-\tilde{\gamma}|\tau|}$, $|\tau| < (1/\varepsilon)e^{\varepsilon|\tau|}$. If $l = 0$ the estimate (8) followed from (5) and the operator $[G_0(\tau, \phi)]$ belongs to the space $C(T_m)$, we therefore suppose that $l > 0$. Consider the difference $[Z_t(\tau, \bar{\phi}, \phi)] = [G_t(\tau, \bar{\phi})] - [G_t(\tau, \phi)]$, where $\bar{\phi} = \phi + \Delta\phi_i e_i$, $e_i^T = (0, \dots, 0, 1, 0, \dots, 0)$ is unit vector and $\Delta\phi_i$ is a scalar constant. $[Z_t(\tau, \bar{\phi}, \phi)]F$ satisfies the matrix equation $(t \neq \tau)$

$$d([Z_t(\tau, \bar{\phi}, \phi)]F)/dt = \Phi_{A,B}\left\{[Z_t(\tau, \bar{\phi}, \phi)]F\right\} + \Psi_{A,B}(t, \tau, \phi, \bar{\phi}), \tag{9}$$

where

$$\begin{aligned}
&\Phi_{A,B}\{X_t\} = A_t(\bar{\phi})X_t - X_t B_t(\bar{\phi}),\\
&\Psi_{A,B}(t, \tau, \phi, \bar{\phi}) = \Phi_{\Delta A, \Delta B}\{[G_t(\tau, \phi)]F\} = \Delta A(\phi_t)([G_t(\tau, \phi)]F) - ([G_t(\tau, \phi)]F)\,\Delta B(\phi_t),\\
&\Delta A(\phi_t) = A(\phi_t(\bar{\phi})) - A(\phi_t(\phi)), \qquad \Delta B(\phi_t) = B(\phi_t(\bar{\phi})) - B(\phi_t(\phi)).
\end{aligned}$$

It has a unique bounded solution on R given by the expression

$$[Z_t(\tau, \bar{\phi}, \phi)]F_t(\phi) = \int_{-\infty}^{\infty} [G_t(s, \bar{\phi})]\Psi_{A,B}(s, \tau, \phi, \bar{\phi})ds. \tag{10}$$

Inequality (5) ensures that the operator-function $[Z_t(\tau, \bar{\phi}, \phi)]$ is bounded on $(-\infty, \infty)$. If we devide the expression (10) on $\Delta\phi_i$ and equal it to zero, we obtain $\lim\limits_{\Delta\phi_i\to 0}(\Delta A(\phi_t))/\Delta\phi_i = D_{\phi_i}A(\phi_t)$, $\lim\limits_{\Delta\phi_i=\to 0}(\Delta B\phi_t)/\Delta\phi_i = D_{\phi_i}B(\phi_t)$, $\lim\limits_{\Delta\phi_i\to 0}([Z_t(\tau, \bar{\phi}, \phi)]F)/\Delta\phi_i = D_{\phi_i}([G_t(\tau, \phi)]F)$. We will be use notations $A(\phi_t(\phi)) = A_{1,t}(\phi)$, $B(\phi_t(\phi)) = A_{2,t}(\phi)$, $\partial/\partial\phi_i = D_{\phi_i}$. Since $\lim\limits_{\Delta\phi\to 0}\Psi_{A,B}(s, \tau, \phi, \bar{\phi})$ uniformly with respect to $\phi \in T_m$ and $\tau, s \in D_2$, it follows that

$$\lim_{\Delta\phi_i\to 0}[Z_t(\tau, \bar{\phi}, \phi)]F/\Delta\phi_i = D_{\phi_i}[G_t(\tau, \phi)]F = \int_{-\infty}^{\infty} J_t(s, \tau, \phi, F)ds, \tag{11}$$

where

$$\begin{aligned}
&J_t(s, \tau, \phi, F) = [G_t(s, \phi)]\Phi_{D_\phi A, D_\phi B}\{[G_s(\tau, \phi)]F\},\\
&\Phi_{D_{\phi_i}A, D_{\phi_i}B}\{[G_s(\tau, \phi)]F\} = D_{\phi_i}A(\phi_s)([G_s(\tau, \phi)]F) - ([G_s(\tau, \phi)]F)D_{\phi_i}B(\phi_s),\\
&D_{\phi_i}A_k(\phi_s) = \sum_{\nu=1}^{m}(\partial A_k(\phi_s(\phi))/\partial(\phi_s)_\nu)(\partial(\phi_s)_\nu/\partial\phi_i), \qquad (k = 1, 2).
\end{aligned}$$

Here D_2 is any bounded domain of the τ, s plane. The value $\lim\limits_{\Delta\phi_i\to 0}[Z_t(\tau, \bar{\phi}, \phi)]$ equal derivative of operator-function, when integral is uniformly convergent.

For the following estimates we will be use the formulas of Faa de Bruno [5]

$$D_\phi^r f(\phi_t(\phi)) = \sum_{q=1}^{r} D_{\phi_t}^q f(\phi_t(\phi)) \sum_p c_{qp}(D_\phi\phi_t(\phi))^{p_1}\left(D_\phi^2\phi_t(\phi)\right)^{p_2}\cdots\left(D_\phi^r\phi_t(\phi)\right)^{p_r}, \tag{12}$$

where $p_1 + p_2 + \dots + p_r = q$, $p_1 + 2p_2 + \dots + rp_r = r$. For obtaining the estimate of function $J_t(s, \tau, \phi, F)$ we need to have the estimate of only the first summand, because the estimate for the second one is different from the first summand by a constant multiplier

$$\|[G_t(s, \phi)]D_{\phi_i}A(\phi_s)[G_s(\tau, \phi)]F\| \le K\exp(-\gamma|s-t| - \gamma|s-\tau| + \tilde{\alpha}|s|)\|F\|. \tag{13}$$

Therefore summarizing the estimates (13) from both summands, we obtain

$$\|J_t(s, \tau, \phi, F)\| \le K\exp(-\gamma|t-s| + \tilde{\alpha}|s| - \gamma|s-\tau|)\|F\|,$$

where $\varepsilon > 0$, $\tilde{\alpha} = \alpha + \varepsilon$, $K = K(\varepsilon)$ and independent of ϕ. Taking $t = 0$, we obtain

$$\|J_0(s, \tau, \phi, F)\| \leq K \exp(-(\gamma - \tilde{\alpha})|s| - \gamma|s - \tau|)\|F\|. \tag{14}$$

For obtaining the estimate of derivative $\|D_{\phi_i}[G_0(\tau, \phi)]\|$ it is necessary to have the estimate of the integral of function $J_0(s, \tau, \phi, F)$, we consider the case $\tau > 0$. We represent the integral as sum of three integrals $(-\infty, \infty) = (-\infty, 0) \bigcup (0, \tau) \bigcup (\tau, \infty)$, after simple transformation we obtain the estimate

$$\int_{-\infty}^{\infty} \|J_0(s, \tau, \phi, F)\| ds \leq K(\varepsilon) e^{-(\gamma - \tilde{\alpha})\tau + \varepsilon\tau} \|F\|,$$

where $K(\varepsilon) = (2/(2\gamma - \tilde{\alpha}) + \tau)$, from which follow estimate

$$||D_{\phi_i}[G_0(\tau, \phi)]|| \leq K(\varepsilon_1) e^{-(\tilde{\gamma} - \alpha)|\tau|}$$

for $\forall \tau \in R$, $\tilde{\gamma} = \gamma - \varepsilon_1$, $\varepsilon_1 = 2\varepsilon$, $K(\varepsilon_1) = K(2/(2\gamma - \tilde{\alpha}) + |\tau|)$. The estimate for the second derivative we obtain from relation

$$D^2_{\phi_i}[G_0(\tau, \phi)]F = \int_{-\infty}^{\infty} D_{\phi_i} J_0(s, \tau, \phi, F) ds. \tag{15}$$

The estimate of both summands of function $D_{\phi_i} J_0(s, \tau, \phi, F)$ will be similar, therefore we need only one of this estimate

$$\begin{aligned} D_{\phi_i} \left([G_0(s, \phi)] D_{\phi_i} A(\phi_s)[G_s(\tau, \phi)]F\right) &= D_{\phi_i}[G_0(s, \phi)] D_{\phi_i} A(\phi_s)[G_s(\tau, \phi)]F \\ &+ [G_0(s, \phi)] D^2_{\phi_i} A(\phi_s)[G_s(\tau, \phi)]F + [G_0(s, \phi)] D_{\phi_i} A(\phi_s) D_{\phi_i}[G_s(\tau, \phi)]F. \end{aligned} \tag{16}$$

For obtaining the estimate of last summand of (16) transform $[G_s(\tau, \phi)]$ to the form $[G_0(\tau - s, \phi_s(\phi))]$, and use the formulas Faa de Bruno

$$D^k_{\phi_i}[G_s(\tau, \phi)] = \sum_{j=1}^{k} D^j_{\phi_i}[G_0(\tau - s, \phi_s(\phi))] \sum_m c_{jm} (D_{\phi_i}\phi_s(\phi))^{m_1} \cdots (D^k_{\phi}\phi_s(\phi))^{m_k}$$

where $m_1 + m_2 + \cdots + m_k = j$, $m_1 + 2m_2 + \cdots + km_k = k$. For $k = 1$ we obtain an estimate

$$\|D_{\phi_i} J_0(s, \tau, \phi, F)\| \leq K(\varepsilon)(e^{-(\tilde{\gamma} - 2\alpha)|s| - \tilde{\gamma}|s - \tau|} + e^{-(\tilde{\gamma} - 2\alpha)|s| - (\tilde{\gamma} - \alpha)|s - \tau|})\|F\|. \tag{17}$$

Taking the integral from expression on right hand side, we obtain estimate

$$\|D^2_{\phi_i}[G_0(\tau, \phi)]F\| \leq K(\varepsilon) e^{-(\tilde{\gamma} - 2\alpha)|\tau|} \|F\|. \tag{18}$$

We carry out the proof by induction. Suppose that inequality (8) holds for $s = k$, we will prove that it then holds for $s = k + 1$. To prove this we differentiate the identity (10) k times, $(t = 0)$

$$D^{k+1}_{\phi_i}[G_0(\tau, \phi)]F = \int_{-\infty}^{\infty} D^k_{\phi_i} J_0(s, \tau, \phi, F) ds. \tag{19}$$

Consider one of summands of function $D^k_{\phi_i} J_0(s, \tau, \phi, F)$, it has the form

$$\sum_{j=0}^{k} C_k^{k-j} ([G_0(s, \phi)] D_{\phi_i} A(\phi_s))^{(k-j)} ([G_0(\tau - s, \phi_s)])^{(j)} F.$$

For the first multiplier, under sign of the sum, an estimate has the form

$$\|D^{k-j}_{\phi_i}([G_0(s, \phi) D_{\phi_i} A(\phi_s))\| \leq \tilde{K} e^{-(\tilde{\gamma} - (k-j+1)\alpha - \varepsilon)|s|}, \tag{20}$$

where $\tilde{K} = K(\varepsilon)\sum\limits_{p=1}^{k-j} C_{k-j}^{k-j-p}$. The Faa de Bruno formulas allow one to obtain an estimate for $[G_0(\tau - s, \phi_s(\phi))]^{(j)}$ of the form

$$\|D^j_{\phi_i}[G_0(\tau - s, \phi_s(\phi))]\| \le jK(\varepsilon)e^{-(\tilde{\gamma}-j\alpha)|s-\tau|+(j\alpha+\varepsilon)|s|}. \tag{21}$$

Using estimate (20), (21) one can obtain estimate

$$\|D^k_{\phi_i}([G_0(s,\phi)]D_{\phi_i}A_q(\phi_s)[G_s(\tau,\phi)])F\| \le \bar{K}_q e^{-(\tilde{\gamma}-(k+1)\alpha)|s|-(\tilde{\gamma}-k\alpha)|s-\tau|}\|F\|,$$

where $\bar{K}_q = K_q(\varepsilon)\sum\limits_{j=1}^{k} jC_k^{k-j}$, $(q = 1, 2)$ independent of ϕ. Summarizing all estimates we have the inequality

$$\|D^{k+1}_{\phi_i}[G_0(\tau,\phi)]F\| \le K(\varepsilon)e^{-(\tilde{\gamma}-(k+1)\alpha)|\tau|}\|F\|$$

and the proof of the Theorem 1 is complete. ■

Theorem 1 allows one to prove the theorem about smoothness of invariant torus of the dichotomous matrix bilinear equation.

Theorem 2. *Let the following conditions be satisfied:* $A(\phi) \in C^l_{\mathrm{Lip}}(T_m)$, $B(\phi) \in C^l_{\mathrm{Lip}}(T_m)$, $a(\phi) \in C^l(T_m)$ *and* $F(\phi) \in C^l(T_m)$, *then the invariant matrix torus (6) of system (1) belongs to the space* $C^l(T_m)$ *and admits the estimate*

$$|U(\phi)|_l \le K|F(\phi)|_l.$$

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