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Nonlinear Diffusion-Convection Systems: Lie and Q-Conditional Symmetries

Roman CHERNIHA † *and Mykola SEROV* ‡

† *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv-4, Ukraine*
E-mail: *cherniha@imath.kiev.ua*

‡ *Dept. of Math., Technical University, 24 Pershotravnevyi Prospekt, Poltava 1, Ukraine*

A class of nonlinear diffusion-convection systems containing two Burgers-type equations is considered. New results of finding Lie and Q-conditional symmetries are presented. Moreover, examples of Lie and non-Lie ansätze and exact solutions of a diffusion-convection system are constructed.

1 Introduction

Nonlinear diffusion-convection (DC) equations of the form

$$U_t = (A(U)U_x)_x + B(U)U_x, \tag{1}$$

where $U = U(t,x)$ is the unknown function, $A(U)$ and $B(U)$ are arbitrary smooth functions and the indices t and x denote differentiation with respect to these variables, generalizes a number of the well known nonlinear second-order evolution equations, describing various processes in physics [1], chemistry [2], biology [3]. The most popular among them is the Burgers equation (BEq)

$$U_t = U_{xx} + \lambda U U_x, \qquad \lambda \in \mathbb{R} \tag{2}$$

arising in several application [4]. Lie symmetry of BEq was found in [5], while the Q-conditional symmetry (i.e., non-classical symmetry [6]) was described in [7] and [8].

In the general case a wide list of Lie symmetries for DC equations of the form (1) is presented in [9]. *A complete description of Lie symmetries,* i.e., group classification of (1) has been done in [10]. The Q-conditional symmetry was also investigated in that paper.

A natural generalization of (1) on several components is the following system of DC equations:

$$\bar{U}_t = (A(\bar{U})\bar{U}_x)_x + B(\bar{U})\bar{U}_x, \tag{3}$$

where $\bar{U} = (U_1, \dots, U_n)$ is the unknown vector function, $A(\bar{U})$ and $B(\bar{U})$ are matrixes $n \times n$ with the elements $a_{ij}(\bar{U})$ and $b_{ij}(\bar{U})$, $i,j = 1,2,\dots,n$ being arbitrary smooth functions. Here we deal with a particular case of (3) at $n = 2$, namely:

$$\begin{aligned} U_t &= \lambda_1 U_{xx} + U U_x + F_1(U,V)V_x, \\ V_t &= \lambda_2 V_{xx} + V V_x + F_2(U,V)U_x, \end{aligned} \tag{4}$$

where $U = U(t,x)$ and $V = V(t,x)$ are unknown functions, while λ_1 and λ_2 are arbitrary constants, F_1 and F_2 are arbitrary smooth functions assumed to be known. It is easily seen that DC system (1) is a coupled system of two Burgers-type equations.

Having in mind a *complete description* of the Lie and Q-conditional symmetries of system (1), which is a very difficult problem in the general case, we now summarize the main results obtained for some subclasses of (1). In Section 2, the complete description of the Lie symmetry of system (1) at $\lambda_1 \neq \lambda_2$ are presented. In the case $\lambda_1 = \lambda_2$ all possible pairs (F_1, F_2) are found when DC system (1) is invariant under the Galilei algebra and its standard extentions. Note that the relevant results for reaction-diffusion systems were obtained in [11, 12, 13].

In Section 3, the determining equations to find the Q-conditional symmetry of system (1) are derived. Furthermore those equations are solved under some assumptions. We have established that system (1) at $F_1 = U + m_1$, $F_2 = V + m_2$, where m_1, m_2 are some constants, admits conditional symmetry operators.

Finally (Section 4), the found symmetries are applied to construct both Lie and non-Lie ansätze of a particular DC system of the form (1). Examples of exact solutions are also presented.

2 Lie symmetry of DC system (1)

It is easily checked that the system (1) is invariant under the operators of time and space translations $P_x = \partial_x$ and $P_t = \partial_t$ for arbitrary functions F_1 and F_2. Following [10], this algebra is called *the trivial Lie algebra* of the system (1). Thus, we aim to find all pairs of functions (F_1, F_2) that lead to extensions of the trivial Lie algebra of this system. Note that we consider only *nonlinear* systems, particularly because linear equations are amenable to numerous classical methods (the Fourier method, method of Laplace transformation and so on).

Now let us formulate a theorem which gives complete information on the classical, i.e., Lie symmetry of the system (1).

Theorem 1. *All possible maximal algebras of invariance (MAI) of the system (1) for any fixed pair (F_1, F_2) and $\lambda_1 \neq \lambda_2$, $\lambda_1\lambda_2 \neq 0$ are presented in Table 1. Any other system of the form (1) with non-trivial Lie symmetry is reduced by the local substitution*

$$x^* = x - mt, \qquad t^* = t, \qquad U^* = U + m, \qquad V^* = V + m, \qquad \lambda \in \mathbb{R} \tag{5}$$

to one of those given in Table 1.

Table 1. MAI of the system (1) at $\lambda_1 \neq \lambda_2$, $\lambda_1\lambda_2 \neq 0$.

/	Nonlinearities	Restrictions	Basic operators of MAI
1.	$F_1 = Uf(\omega)$ $F_2 = Vg(\omega)$	$\omega = U/V$	P_t, P_x $D = 2tP_t + xP_x - U\partial_U - V\partial_V$
2.	$F_1 = f(\omega)$ $F_2 = g(\omega)$	$\omega = U - V$	P_t, P_x $G_x = tP_x - (\partial_U + \partial_V)$
3.	$F_1 = \alpha_1(U - V)$ $F_2 = \alpha_2(V - U)$	$\alpha_1 \neq 0$ or $\alpha_2 \neq 0$	P_t, P_x, G_x, D
4.	$F_1 = 0$ $F_2 = 0$		P_t, P_x, G_x, D $\Pi = tD - t^2P_t - x(\partial_U + \partial_V)$

The proof of Theorem 1 is based on the classical Lie scheme (see, e.g., [15, 14]) and is non-trivial because the system (1) contains two arbitrary functions of two variables. The proof of this and following theorems will be published in [16]).

Remark 1. Cases 3 and 4 in Table 1 are natural prolongations of case 2, because the extended Galilei algebra $AG_1^0(1,1) = \langle P_t, P_x, G_x^0, D\rangle$ and the generalized Galilei algebra $AG_2^0(1,1) = \langle P_t, P_x, G_x^0, D, \Pi\rangle$ are known to be standard extensions of the Galilei algebra $AG^0(1,1) = \langle P_t, P_x, G_x^0\rangle$ with zero mass (for details see [11, 12, 15]).

It turns out that the case $\lambda_1 = \lambda_2 \neq 0$ (without losing generality we can put $\lambda_1 = 1$) is more complicated than the case considered above and its complete description will be done in [16]. Here the most interesting cases are only presented.

Theorem 2. *In the case $\lambda_1 = \lambda_2 = 1$, DC system (1) for $F^1_V \neq 0$ or $F^2_U \neq 0$ is invariant under the Galilei algebra if and only if*

$$F^k = \phi(\omega) - (-1)^k \frac{U-V}{2}, \qquad k = 1,2, \quad \omega = (U-V)^\gamma \exp(U+V), \quad 0 \neq \gamma \in \mathbb{R},$$

where ϕ is an arbitrary function. The corresponding basic operators of the Galilei algebra are

$$P_t, \quad P_x, \quad G^0_x = t\partial_x + \frac{U-V}{\gamma}(\partial_U - \partial_V) - (\partial_U + \partial_V).$$

Theorem 3. *In the case $m_1 \neq m_2 \in \mathbb{R}$, MAI of nonlinear DC system*

$$\begin{aligned} &U_t = U_{xx} + UU_x + (m_1 + U)V_x, \\ &V_t = V_{xx} + VV_x + (m_2 + V)U_x \end{aligned} \tag{6}$$

is the generalized Galilei algebra $AG^0_2(1,1)$ with zero mass generated by the basic operators

$$\begin{aligned} &P_t, \quad P_x, G^0_x = tP_x + Q^0_1, \quad D_0 = 2t\partial_t + x\partial_x - U\partial_u - v\partial_v + Q^0_2, \\ &\Pi_0 = tD_0 - t^2\partial_t + xQ^0_1 + \frac{2}{m_1 - m_2}(\partial_U - \partial_V). \end{aligned} \tag{7}$$

In the case $m_1 = m_2 = 0$, MAI of (6) is infinite-dimensional algebra generated by the operators

$$\begin{aligned} &P_t, \quad P_x, \quad Q_1 = \frac{1}{2}(U-V)(\partial_U - \partial_V), \quad G_x = t\partial_x + \frac{x}{2}Q_1 - Q_2, \\ &D = 2t\partial_t + x\partial_x + \frac{1}{2}Q_1 - (U\partial_U + V\partial_V), \quad \Pi = tD_1 - t^2\partial_t + \frac{x^2}{4}Q_1 - xQ_2, \end{aligned} \tag{8}$$

which form the $AG_2(1,1)$ with non-zero mass, and the operator

$$X^\infty = (MU + MV - 2M_x)(\partial_U - \partial_V), \tag{9}$$

where $M = M(t,x)$ is an arbitrary solution of the linear diffusion equation $M_t = M_{xx}$.

In formulas (7) and (8) the operators

$$\begin{aligned} &Q^0_1 = \frac{U+V+2m_1}{m_2 - m_1}\partial_U + \frac{U+V+2m_2}{m_1 - m_2}\partial_V, \\ &Q^0_2 = \frac{m_2U + m_1V + 2m_1m_2}{m_2 - m_1}(\partial_U - \partial_V) + U\partial_U + V\partial_V, \qquad Q_2 = \frac{1}{2}(\partial_U + \partial_V). \end{aligned}$$

Remark 2. In the case $m_1 = m_2 \neq 0$, system (6) is reduced to the same with $m_1 = m_2 = 0$ by the local substitution (5).

3 *Q*-conditional symmetry of DC system (1)

In this section we study Q-conditional symmetry of nonlinear DC system (1). Nevertheless the main idea of the notion of Q-conditional symmetry (non-classical symmetry) is very simple and was introduced by Bluman and Cole more than 30 years ago [6], it is a very non-trivial problem to find new operators of Q-conditional symmetry for nonlinear equations arising in applications.

Moreover, to our knowledge there are even no examples of operators of Q-conditional symmetry in the case of *DC systems* of the form (3).

We remind the reader that every operator of Lie symmetry is also a Q-conditional symmetry operator therefore hereinafter we will find only purely conditional symmetry operators. It is worth also reminding on the following property of such operators: if the operator

$$Q = \partial_t + \xi(t,x,U,V)\partial_x + \eta^1(t,x,U,V)\partial_U + \eta^2(t,x,U,V)\partial_V, \tag{10}$$

where the ξ, η^1 and η^2 being the known functions, is one of the Q-conditional symmetry for DC system (1) then the operator $N(t,x,U,V)\,Q$ being N an arbitrary nonvanishing function is also the Q-conditional symmetry operator. Thus we will seek only operators of the canonical form (10). Of course, one can also find Q-conditional symmetry operators of the canonical form

$$Q = \partial_x + \eta^1(t,x,U,V)\partial_U + \eta^2(t,x,U,V)\partial_V,$$

however we aim to discuss such possibility elsewhere.

Using the known procedure (see, for example, [15], chapter 5) to construction of the operators Q of the form (10), where the coefficients ξ, η^1 and η^2 must be found, we have established the following theorem.

Theorem 4. *DC system (1) is Q-conditional invariant under the operator (10), if and only if the functions ξ, η^1, η^2 satisfy the following determining equations:*

$$\xi_{UU} = \xi_{VV} = \xi_{UV} = 0, \tag{11}$$

$$\lambda_1\eta_{VV} + F^1\xi_V = 0, \qquad \lambda_2\eta^2_{UU} + F^2\xi_U = 0, \tag{12}$$

$$\begin{aligned} &\lambda_1\eta^1_{UU} - 2\lambda_1\xi_{xU} + 2(\xi+U)\xi_U + \frac{\lambda_1}{\lambda_2}F^2\xi_V = 0,\\ &\lambda_2\eta^2_{VV} - 2\lambda_2\xi_{xV} + 2(\xi+V)\xi_V + \frac{\lambda_2}{\lambda_1}F^1\xi_U = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} &2\lambda_1\eta^1_{UV} - 2\lambda_1\xi_{xV} + \frac{1}{\lambda_2}(\lambda_2U + \lambda_1V + (\lambda_1+\lambda_2)\xi)\xi_V + 2F^1\xi_U = 0,\\ &2\lambda_2\eta^2_{UV} - 2\lambda_2\xi_{xU} + \frac{1}{\lambda_1}(\lambda_2U + \lambda_1V + (\lambda_1+\lambda_2)\xi)\xi_U + 2F^2\xi_V = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &\lambda_1\eta^1_{xx} - \eta^1_t - 2\xi_x\eta^1 + \left(\frac{\lambda_1}{\lambda_2} - 1\right)\eta^2\eta^1_V + U\eta^1_x + F^1\eta^2_x = 0,\\ &\lambda_2\eta^2_{xx} - \eta^2_t - 2\xi_x\eta^2 + \left(\frac{\lambda_2}{\lambda_1} - 1\right)\eta^1\eta^2_U + V\eta^2_x + F^2\eta^1_x = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &\lambda_1(2\eta^1_{xU} - \xi_{xx}) + (2\xi+U)\xi_x - 2\eta^1\xi_U + \left(1 - \frac{\lambda_1}{\lambda_2}\right)\eta^2\xi_V - \frac{\lambda_1}{\lambda_2}F^2\eta^1_V\\ &\qquad + F^1\eta^2_U + \xi_t + \eta^1 = 0,\\ &\lambda_2(2\eta^2_{xV} - \xi_{xx}) + (2\xi+V)\xi_x - 2\eta^2\xi_V + \left(1 - \frac{\lambda_2}{\lambda_1}\right)\eta^1\xi_U - \frac{\lambda_2}{\lambda_1}F^1\eta^2_U\\ &\qquad + F^2\eta^1_V + \xi_t + \eta^2 = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} &2\lambda_1\eta^1_{xV} + (\xi_x - \eta^1_U + \eta^2_V)F^1 - 2\eta^1\xi_V\\ &\qquad + \frac{1}{\lambda_2}[(\lambda_2-\lambda_1)\xi + \lambda_2U - \lambda_1V]\eta^1_V + \eta^1F^1_U + \eta^2F^1_V = 0,\\ &2\lambda_2\eta^2_{xU} + (\xi_x - \eta^2_V + \eta^1_U)F^2 - 2\eta^2\xi_U\\ &\qquad + \frac{1}{\lambda_1}[(\lambda_2-\lambda_1)\xi + \lambda_2U - \lambda_1V]\eta^2_U + \eta^2F^2_V + \eta^1F^2_U = 0. \end{aligned} \tag{17}$$

The overdetermined system of nonlinear equations (11)–(17) is very complicated and we have not constructed its general solutions. On the other hand, it is possible to construct the general solution under the additional condition $\eta^1_t = \eta^2_t = \eta^1_x = \eta^2_x = 0$, i.e., assuming $\eta^k = \eta^k(U, V)$, $k = 1, 2$. Under such assumption the subsystem (15) with $\lambda_1 = \lambda_2$ is reduced to the condition $\xi_x = 0$ therefore other subsystems can be easily solved.

Theorem 5. *DC system (1) is Q-conditional invariant under the operator*

$$Q = \partial_t + \xi^1(t, x, U, V)\partial_x + \eta^1(U, V)\partial_U + \eta^2(U, V)\partial_V, \tag{18}$$

if and only if

$$\lambda_1 = \lambda_2 = 1, \qquad F^1 = m_1 + U, \qquad F^2 = m_2 + V, \qquad m_1, m_2 \in \mathbb{R} \tag{19}$$

and then the coefficients of the operator (18) have the form

$$\begin{aligned}
\xi^1 &= \frac{1}{2}(U + V) + \alpha_0, \\
\eta^1 &= -\frac{1}{4}\left[U(U + V)^2 + 2\alpha_0 U(U + V)\right] + \beta_0 U + \gamma_1, \\
\eta^2 &= -\frac{1}{4}\left[V(U + V)^2 + 2\alpha_0 V(U + V)\right] + \beta_0 V + \gamma_2,
\end{aligned} \tag{20}$$

if $m_1 = m_2 = 0$, *and*

$$\begin{aligned}
\xi^1 &= \frac{1}{2}(U + V), \\
\eta^1 &= -\frac{1}{4}\left[(U + m_1)(U + V)^2 + (m_2 - m_1)U^2\right] + \beta_0 U + \gamma_1, \\
\eta^2 &= -\frac{1}{4}\left[(U + m_2)(U + V)^2 + (m_1 - m_2)V^2\right] + \beta_0 V + \gamma_2,
\end{aligned} \tag{21}$$

if $m_1 \neq m_2$. *Here* α_0, β_0, γ_1, γ_2 *are arbitrary constants.*

One can see that the above listed additional conditions on the form of the operator Q are very strong because they lead only to the fixed nonlinearity $F_1 = U + m_1$, $F_2 = V + m_2$. The next theorem illustrates that the requirement $\lambda_1 = \lambda_2$ is very important.

Theorem 6. *DC system (1) at*

$$\lambda_1 \neq \lambda_2, \qquad F^1 = \frac{\lambda_1}{\lambda_2}(U + m), \qquad F^2 = \frac{\lambda_2}{\lambda_1}(V - m), \qquad m \in \mathbb{R} \tag{22}$$

is invariant under the trivial Lie algebra generated by the basic operators P_t *and* P_x *while one admits the operator of the Q-conditional symmetry*

$$Q = \partial_t - m\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\partial_x + \frac{U + V}{(\lambda_1 - \lambda_2)t}(\lambda_1\partial_U - \lambda_2\partial_V). \tag{23}$$

4 Ansätze and exact solutions of a DC system

In this section we shall deal with the nonlinear DC system (6). It follows from Theorem 3 that MAI of (6) for $m_1 \neq m_2$ is the generalized Galilei algebra $AG^0_2(1, 1)$ with the basic operators (7). It seems reasonable to construct Lie ansätze and to seek exact solutions of system (6) using operators (7). A full set of non-equivalent (non-conjugate) one-dimensional subalgebras of the $AG_2(1, 1)$ algebra is well-known [14]. Taking into account the similarity of structures of the

$AG_2(1,1)$ algebra and $AG_2^0(1,1)$ algebra, a full set of non-equivalent one-dimensional subalgebras of the $AG_2^0(1,1)$ algebra was also constructed, namely:

$$k_1\partial_t + k_2\partial_x, \quad \partial_t + k_3 G, \quad D, \quad \partial_t + \Pi, \tag{24}$$

where k_1, k_2, k_3 are arbitrary constants. Let us apply each of them for reduction of system (6) to systems of ordinary differential equations (ODEs).

a) The operator $k_1\partial_t + k_2\partial_x$ generates the ansatz

$$U = \varphi(\omega), \qquad V = \psi(\omega), \qquad \omega = k_2 t - k_1 x, \tag{25}$$

where φ, ψ are unknown functions. Substituting (25) into system (6), we arrive at the ODEs system

$$\begin{aligned} &k_2\dot\varphi = k_1^2\ddot\varphi - k_1\varphi\dot\varphi - k_1(\varphi + m_1)\dot\psi,\\ &k_2\dot\psi = k_1^2\ddot\psi - k_1(\psi + m_2)\dot\varphi - k_1\psi\dot\psi, \end{aligned} \tag{26}$$

(hereinafter $\dot\varphi = \frac{d\varphi}{d\omega}$, $\ddot\varphi = \frac{d^2\varphi}{d\omega^2}$).

b) The operator $\partial_t + k_3 G$ generates the ansatz

$$\begin{aligned} &U = \frac{(k_3 t - m_1)\varphi(\omega) + \psi(\omega) + (k_3 t - m_1)^2}{m_1 - m_2} - m_1,\\ &V = \frac{(k_3 t - m_2)\varphi(\omega) + \psi(\omega) + (k_3 t - m_2)^2}{m_2 - m_1} - m_2, \qquad \omega = x - \frac{k_3}{2}t^2, \end{aligned} \tag{27}$$

which reduces system (6) the ODEs system

$$\begin{aligned} &\ddot\varphi - \varphi\dot\varphi + \dot\psi - 2k_3 = 0,\\ &\ddot\psi - \psi\dot\varphi - k_3\varphi = 0. \end{aligned} \tag{28}$$

c) The operator D generates the ansatz

$$\begin{aligned} &U = \frac{m_1 t^{-1/2}\varphi(\omega) + t^{-1}\psi(\omega) + m_1 m_2}{m_1 - m_2},\\ &V = \frac{m_2 t^{-1/2}\varphi(\omega) + t^{-1}\psi(\omega) + m_1 m_2}{m_2 - m_1}, \qquad \omega = t^{-1/2}x. \end{aligned} \tag{29}$$

which reduces system (6) the ODEs system

$$\begin{aligned} &\ddot\varphi + \varphi\dot\varphi + \frac{1}{2}(\omega\dot\varphi + \varphi) - \dot\psi = 0,\\ &\ddot\psi + \psi\dot\varphi + \frac{1}{2}\omega\dot\psi + \psi = 0. \end{aligned} \tag{30}$$

d) Finally, the operator $\partial_0 + \Pi$ generates the ansatz

$$\begin{aligned} U = {} & \frac{1}{m_1 - m_2}\Big\{ \left(t^2+1\right)^{-1/2} m_1(\varphi(\omega) - 2t\omega)\\ & - \left(t^2+1\right)^{-1}(\psi(\omega) + t\omega\varphi(\omega) - 2t) + \omega^2 + m_1 m_2\Big\},\\ V = {} & \frac{1}{m_2 - m_1}\Big\{ \left(t^2+1\right)^{-1/2} m_2(\varphi(\omega) - 2t\omega)\\ & - \left(t^2+1\right)^{-1}(\psi(\omega) + t\omega\varphi(\omega) - 2t) + \omega^2 + m_1 m_2\Big\}, \qquad \omega = \left(t^2+1\right)^{-1/2} x, \end{aligned} \tag{31}$$

which reduces system (6) the ODEs system

$$\ddot{\varphi} + \varphi\dot{\varphi} + \dot{\psi} = 0,$$
$$\ddot{\psi} + \psi\dot{\varphi} = \omega(\omega\dot{\varphi} + \varphi). \tag{32}$$

Having solutions of the ODEs systems (26), (28), (30), (32) and using the relevant ansätze one easily constructs solutions of the original nonlinear DC system (6). For example, a particular solution of system (28) leads to the following exact solution of system (6):

$$U = \frac{1}{m_1 - m_2}\left(\frac{x^2 - 2m_1tx - 4t}{t^2+1} + \frac{12}{x^2} + \frac{6m_1}{x} + m_1m_2\right),$$
$$V = \frac{1}{m_2 - m_1}\left(\frac{x^2 - 2m_2tx - 4t}{t^2+1} + \frac{12}{x^2} + \frac{6m_2}{x} + m_1m_2\right). \tag{33}$$

By means of the known technique (see for details [11, 15]) for the continuous transformations generated by the basic operators (7), solution (33) can be multiplied to a five-parameter family of solutions. Such multiplication is possible for any given solution of system (6). In particular case, using transformations generated by the Galilei operator G^0_x, any *time-independent (stationary) solution* $(U_0(x), V_0(x))$ is converted to the following one-parameter family of solutions of system (6)

$$U = U_0(x+\epsilon t) - \epsilon\frac{U_0(x+\epsilon t) + V_0(x+\epsilon t) + 2m_1}{m_2 - m_1},$$
$$V = V_0(x+\epsilon t) + \epsilon\frac{U_0(x+\epsilon t) + V_0(x+\epsilon t) + 2m_2}{m_2 - m_1}, \tag{34}$$

where ϵ is an arbitrary real parameter.

Let us apply the Q-conditional symmetry operators for the construction of ansätze and exact solutions of system (6). It follows from Theorem 5 that system (6) for $m_1 \neq m_2$ is Q-conditional invariant with respect to the operator

$$Q = \partial_t + \frac{U+V}{2}\partial_x - \frac{1}{4}\Big\{(U+V)^2(U+m_1)\partial_U$$
$$+ (U+V)^2(V+m_2)\partial_V - (m_1-m_2)\left(U^2\partial_U - V^2\partial_V\right)\Big\}. \tag{35}$$

To construct the relevant solutions of system (6), it is necessary to integrate the Lagrange system

$$\frac{dt}{-4} = \frac{dx}{-2(U+V)} = \frac{dU}{(U+m_1)(U+V)^2 + (m_2-m_1)U^2}$$
$$= \frac{dV}{(V+m_2)(U+V)^2 + (m_1-m_2)V^2}. \tag{36}$$

In contrast to the analogous systems for Lie operators (24), system (36) is *nonlinear* with respect to the unknown functions U and V, therefore there is a problem to construct its general solution. It turns out that this system can be essentially simplified by the substitution

$$t = t, \qquad x = x, \qquad w = U+V, \qquad z = \frac{m_1-m_2}{2}\left(\frac{U+V}{U-V} - \frac{m_1+m_2}{m_1-m_2}\right). \tag{37}$$

Indeed, the relevant calculations show that system (36) takes the form

$$\frac{dt}{-4} = \frac{dx}{-2w} = \frac{dw}{w^2(w-2z)} = \frac{dz}{w\,(z^2 - m_1m_2)}. \tag{38}$$

The first integrals J_1, J_2, J_3 of system (38) depend on the sign of the term m_1m_2, i.e., there are three different cases: $m_1m_2 = 0$, $m_1m_2 > 0$ and $m_1m_2 < 0$. Considering the first of them (other two cases see in [16]), we obtain

$$J_1 = t + \frac{4}{wz} - \frac{2}{z^2}, \qquad J_2 = x - \frac{2}{z}, \qquad J_3 = \frac{3}{wz^2} - \frac{1}{z^3}. \tag{39}$$

Thus, we construct the non-Lie ansatz (6)

$$J_1 = \varphi(J_2), \qquad J_3 = \psi(J_2), \tag{40}$$

being φ and ψ new unknown functions, for finding solutions of the original nonlinear DC system (6). Substituting ansatz (40) into (6) in the case $m_2 = 0$, $m_1 = 1$ (this system for $m_1 \neq 1$ is reduced to the same with $m_1 = 1$ by the substitution $t \to m_1^{-2}t$, $x \to m_1^{-1}x$, $U \to m_1U$, $V \to m_1V$), we arrive at the ODEs system

$$\begin{aligned} &\ddot{\varphi} + 1 = 0, \\ &4\dddot{\psi} + \dot{\varphi} = 0. \end{aligned} \tag{41}$$

Since (41) is the linear system, its general solution can be easily found. Thus, substituting one into ansatz (39), (40), we obtain the two-parameter family of solutions of system (6) with $m_2 = 0$, $m_1 = 1$:

$$U = \frac{\frac{2}{3}x^3 + 2x^2 + 4C_1(x+2) + 4(C_2 - t)}{W}, \qquad V = \frac{4(t - C_2) - 2x^2}{W}, \tag{42}$$

where $W = \frac{1}{12}x^4 + t^2 + C_1(x^2 - 2t) + 2C_2x$ and C_1, C_2 are arbitrary parameters.

Some other non-Lie ansätze and exact solutions are presented in [16].

5 Conclusions

In this paper, Theorem 1 is presented that gives a complete description of Lie symmetries of the nonlinear diffusion-convection system (1) for $\lambda_1 \neq \lambda_2$, $\lambda_1\lambda_2 \neq 0$. In contrast to reaction-diffusion systems (a complete description of Lie symmetries of those systems was done in [13]), we have established only four non-equivalent cases when system (1) is invariant with respect to the non-trivial Lie algebras. Obviously, the nonlinear fixed terms UU_x and VV_x (see (1)) play a role of the strong restrictions of Lie symmetry for system (1).

The nonlinear DC system (6) with unique symmetry properties has been also found. This system is invariant under the generalized Galilei algebras $AG_2^0(1,1)$ in the case $m_1 \neq m_2$ and $AG_2(1,1)$ in the case $m_1 = m_2$ (see Theorem 3). On the other hand, system (6) admits the operators of Q-conditional symmetry with the cubic nonlinearities on the dependent variables U and V (see Theorem 5). To our knowledge, such operators for *system of nonlinear evolution equations* are found for the first time. Analogous operators were found before for single reaction-diffusion equations [17, 15, 18] and single reaction-diffusion-convection equations [10]. Finally, it should be stressed that the process of reduction of (6) is very non-trivial if one uses the Q-conditional symmetry operators (18), (20)–(21). However, the relevant reduction leads to very simple ODEs systems (see, for example, (41) that were easily solved therefore exact solutions of the nonlinear DC system (6) were obtained.

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