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C^*-Algebras Associated with $\mathcal{F}_{2^n}$ Zero Schwarzian Unimodal Mappings

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In this paper we consider C^*-algebras connected with a simple unimodal non-bijective dynamical system (f, I) with zero Schwarzian. We associate with f a C^*-algebra $C^*(\mathcal{A}_f)$. In the first part we describe the dynamics of (f, I). In the second part we describe the set of irreducible representation of $C^*(\mathcal{A}_f)$ for a special subclass of mappings (Theorem 3) and give realization (Theorem 4) of this algebra as C^*-algebra generated by continuous fields of C^*-algebras on the spectrum of $C^*(\mathcal{A}_f)$. As a result we find out when two such C^*-algebras are isomorphic.

1 Zero Schwarzian unimodal mappings

Many important examples of C^*-algebras arising in physical models are connected with dynamical systems. In particular, the two-parameter unit quantum disk algebra [1] is generated by the relation

$$\begin{aligned}&qzz^* - z^*z = q - 1 + \mu(1 - zz^*)(1 - z^*z),\\ &0 \le \mu \le 1, \qquad 0 \le q \le 1, \qquad (\mu, q) \ne (0, 1),\end{aligned}$$

which can be rewritten [2] in the form $XX^* = F(X^*X)$, where

$$F(\lambda) = \frac{(q + \mu)\lambda + 1 - q - \mu}{\mu\lambda + 1 - \mu}.$$

In present paper we investigate unimodal deformation of the above relation. Consider a continuous unimodal map $f : [0, 1] \to [0, 1]$ with zero Schwarzian that consists of two hyperbolae:

$$f(x) = \begin{cases} f_1(x) = \dfrac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}, & x \in [0, \rho],\\[2mm] f_2(x) = \dfrac{\alpha_2 x + \beta_2}{\gamma_2 x + \delta_2}, & x \in (\rho, 1].\end{cases}$$

Let $\mathrm{Orb}_+(f)$ be a set of all non-cyclic positive orbits [7]. Considering mappings up to topological conjugacy [3] we can assume that $\gamma_2 = 0$, $\delta_2 = 1$. In the present paper we restrict ourselves with the following types of f (see Fig. 1):

Type 1: $f_2(1) = 1$, $f_1(\rho) = f_2(\rho) = 0$, Type 2: $f_2(1) = 0$, $f_1(\rho) = f_2(\rho) = 1$,

$$f(x) = \begin{cases} f_1(x) = \dfrac{\alpha x - \alpha\rho}{\gamma x + \delta}, & x \in [0, \rho],\\[2mm] f_2(x) = \dfrac{x - \rho}{1 - \rho}, & x \in (\rho, 1];\end{cases} \qquad f(x) = \begin{cases} f_1(x) = \dfrac{\alpha(x - \rho) + \delta + \gamma\rho}{\gamma x + \delta}, & x \in [0, \rho],\\[2mm] f_2(x) = \dfrac{x - 1}{\rho - 1}, & x \in (\rho, 1].\end{cases}$$

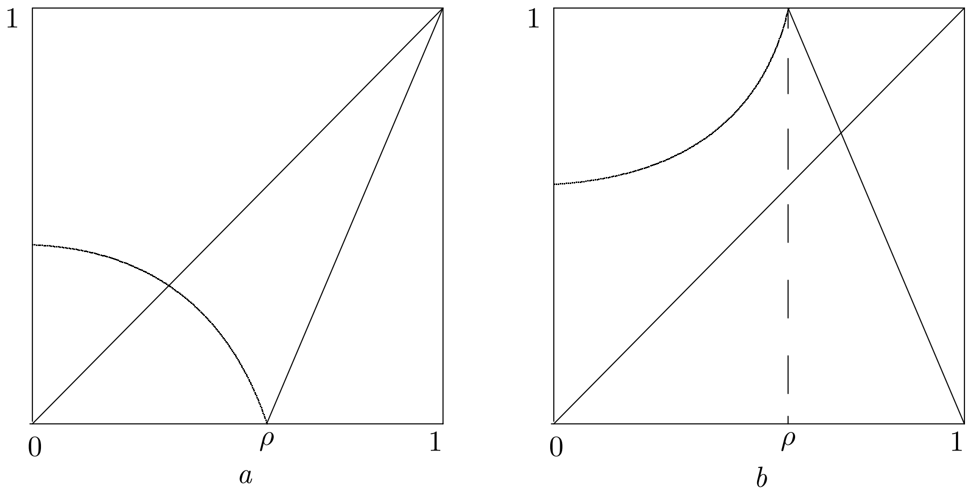

Figure 1.

Lemma 1. *Let (f, I) be $\mathcal{F}_{2^n}$ dynamical system. Then $n \leqslant 3$ and for each $i \in \{0, 1, 2\}$ only two following cases are possible.*

1. There exists only one attractive cycle of the period 2^i, smaller cycles are repellent and no cycles of larger periods.

2. There exists an interval of periodic points such that the middle point of the interval has period 2^i, other points of the interval have period 2^{i+1}, smaller cycles are repellent and no cycles of larger periods.

Cases for $i = 0$ correspond only to type 1. Cases for $i = 1$ correspond either to type 1 or to type 2. Cases for $i = 2$ correspond only to type 2.

Proof. First consider type 1 mappings. Define ρ and ρ_1 as $f_1(0) = \rho_1$ and $f_1(\rho) = 0$. Let x_0 be a stable point of $f_1(x)$ that lies between 0 and ρ. It can be easily checked that: $\mathrm{Sign}\,(1 - |f'(x_0)|) = \mathrm{Sign}\,(\rho - \rho_1)$.

When f has an attracting stable point, $|f_1'(x_0)|$ is less than 1. Hence $\rho > \rho_1$. Therefore $\forall\ x \in [0, x_0)\ f^{(2)}(x) > x$, $\forall\ x \in (x_0, 1)\ f^{(2)}(x) < x$ and mapping f has no cycles of the period two. We will observe the same situation until $|f_1'(x_0)|$ equals 1. When $|f_1'(x_0)| = 1$ the left hyperbola is symmetric with respect to diagonal. Therefore each point of the interval $[0, \rho]$ except x_0 has period two. As follows from a simple geometrical considerations in this case mapping f has no cycles of period four. If $|f_1'(x_0)|$ is more than 1 or equivalently the stable point becomes repellent, then two following cases are possible: 1) any cycle of period 2^n exists, 2) there exists either attracting cycle of period two or an interval of periodic points such that the middle point of the interval has period two and other points have period four. Therefore in the first case the dynamical system is not $\mathcal{F}_{2^n}$. In the second case the dynamical system has obviously no cycles of larger periods.

Let us prove the latter statement.

1. The proof is by induction on n. The base of induction is existence of repellent cycles of periods one and two. Let $\gamma \neq 0$. Hence we can put $\gamma = 1$. When $\gamma = 0$, the proof is trivial because if cycle of period 2^n exists, then one is obviously repellent. Let x_n be first from the left stable point of $f^{(2^n)}$. It is really uninteresting work to show that if a cycle of period two is repellent, then for all $x \in [0, x_2]$ $|f^{(4)^{'}}(x)| > 1$. Hence we can add this fact to the base of induction. Let x' be a point such that $f_1(x') = \rho$. It is also very boring to show that for any f_1 such that x_0 is repellent point the derivative $(f_1(f_1(x)))'$, $x \in [x', x_0]$ is more than one.

Now we prove that if the cycles of periods 2^{n-1} and 2^n are repellent for some $n \geqslant 1$ and $\forall\ x \in [0, x_n]$ $|f^{(2^n)^{'}}(x)| > 1$, then there exists a cycle of period 2^{n+1} which is repellent and $\forall\ x \in [0, x_{n+1}]$ $|f^{(2^{n+1})^{'}}(x)| > 1$.

1) First we prove that if a stable point x_0 is repellent, then there exists a cycle of period two. If point x_0 is repellent, then $\rho < \rho_1$. Therefore $f^{(2)}(0) > 0$ and exists $x_1 \in [0, x_0]$ such that $f(x_1) = \rho$ and $f^{(2)}(x_1) = 0$. Hence there exists $x_2 \in [0, x_1]$ such that $f^{(2)}(x_2) = x_2$.

Now consider $f^{(2^n)}$. It is evident that $(f^{(2^n)}, J)$, where $J = [0, x_{n-1}]$ and x_{n-1} is the first from the left stable point of $f^{(2^{n-1})}$, is equivalent to type 1 mapping. Therefore $f^{(2^n)}$ has cycle of period two or equivalently f has cycle of period 2^{n+1}.

2) Now let repellent cycles of the periods 2^{n-1} and 2^n exist. Consider a dynamical system $(f^{(2^n)}, J)$, where $J = [0, x_{n-1}]$. It is clearly type one system for some $\tilde{\rho}$, $\tilde{f}_1$ and $\tilde{f}_2$. By induction hypothesis $\forall\ x \in [0, x_n]$ $|\tilde{f}_1^{\,'}(x)| > 1$. Also $\forall\ x \in [\tilde{\rho}, x_{n-1}]$ $\tilde{f}_2^{\,'}(x) > 1$. Therefore, for all $x \in [0, x_{n+1}]$ $|f^{(2^{(n+1)})'}(x)| = |(\tilde{f}_2(\tilde{f}_1(x)))'| > 1$.

Now consider type 2 mapping. Let $g(x) = f(f(x))$ (see Fig. 2). We will consider $g(x)$ in $[0, s] \times [0, s]$, where s is a stable point of f, bearing in mind parallel considerations for the right corner. Like in case one we define ρ and ρ_1 as $g(0) = \rho_1$ and $g(\rho) = 0$. Let x_0 be a stable point of $g(x)$ that lies between 0 and ρ. Thus we obtain a situation of the type 1. Therefore if (f, I) is $\mathcal{F}_{2^n}$, then for each $i = 1, 2$ mapping f has either an attracting cycle of period 2^i or an interval of periodic points such that the middle point of the interval has period 2^i and other points have period 2^{i+1}. ■

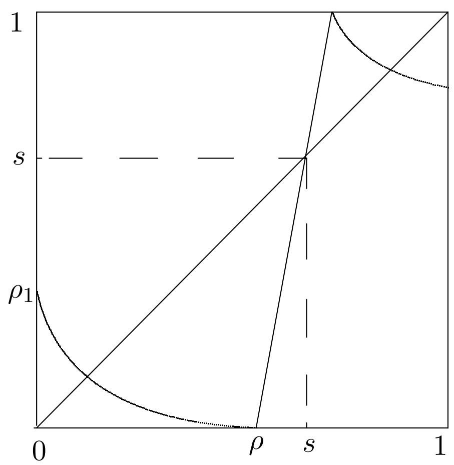

Figure 2.

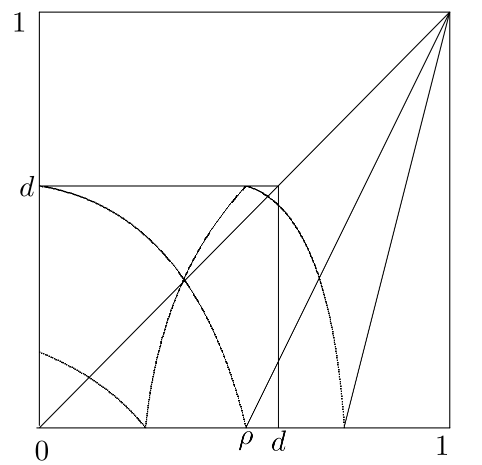

Figure 3.

Theorem 1. *Let (f, I) be type 1 or type 2 dynamical system. Let s be its stable repellent point and β_j its repellent cycle of period 2^j (if it exists). For type one mapping $j = 0$ and $\beta_0 \neq s$. For type two mapping $j = 1$. Define $P_s = \{\delta | \delta \in \mathrm{Orb}_+(f),\ \alpha(\delta) = s\}$ and $P_{\beta_j} = \{\delta | \delta \in \mathrm{Orb}_+(f),\ \alpha(\delta) = \beta_j\}$. Let i be defined as in Lemma 1. Then*

I. For dynamical system of type 1 and $i = 0$ or for dynamical system of type 2 and $i = 1$:

- *1) $\mathrm{Orb}_+(f) = P_s$;*
- *2) there exists $I_s = [t_1, t_2)$ and one-to-one mapping $\phi : I_s \to P_s$ such that $t \in \phi(t)$ for every $t \in I_s$;*
- *3) I_s can be chosen to lie in arbitrary neighborhood of s.*

II. For dynamical system of type 1 and $i = 1$ or for dynamical system of type 2 and $i = 2$:

- *1) $\mathrm{Orb}_+(f) = P_s \dot{\cup} P_{\beta_j}$;*
- *2) there exists $I_{\beta_j} = [t_1, t_2)$ and one-to-one mapping $\phi : I_{\beta_j} \to P_{\beta_j}$ such that $t \in \phi(t)$ for every $t \in I_{\beta_j}$. There exists $I_s = [t_1, t_2)$ and one-to-one mapping $\phi : I_s \to P_s$ such that $t \in \phi(t)$ for every $t \in I_s$;*
- *3) $I_s \cap I_{\beta_j} = \varnothing$. Moreover I_{β_j} can be chosen to lie in arbitrary neighborhood of β_j.*

Proof. Let us first consider type 1 mapping. Define ρ and ρ_1 as in lemma: $f_1(0) = \rho_1$ and $f_1(\rho) = 0$. By the lemma $\rho \geqslant \rho_1$. Define intervals $I_n = f_2^{(-n)}([0,\rho))$, $n \geqslant 1$. Note that $I_i \cap I_j = \varnothing$ $i \neq j$, $\forall$ n $I_n \cap [0,\rho) = \varnothing$ and $\cup_{n\geqslant 1} I_n \cup [0,\rho) = [0,1)$. It is easy to see that for $x \in \delta \in \mathrm{Orb}_+(f)$, $x \in I_n$, $\quad f^{-1}(x) = f_2^{-1}(x) \in I_{n+1}$ and $\alpha(x) = s$. Now prove that any of the intervals I_n can be chosen as I_s. Since $\forall\, x \in I_n$ $f^{(-n)}(x) \notin I_n$ we obtain that different points of I_n correspond to different trajectories. It is clear to see that if $x \in \delta \in \mathrm{Orb}_+(f)$ and $x \in I_i$, then for any j there exists point $y \in \delta$ such that $y \in I_j$. Now let $\exists$ $x \in \delta \in \mathrm{Orb}_+(f)$, $x \in [0,\rho)$ and $x \notin \mathrm{Per}(f)$. The stable point $x_0 \in [0,\rho)$ is repellent for f_1^{-1} and therefore there exists a natural number m such that $f_1^{(-m)}(x) \in [\rho_1,\rho)$ and $f^{(-1)}(f_1^{(-m)})(x) = f_2^{-1} f_1^{(-m)}(x) \in I_1$. By $f^{-1}(x)$ we mean either $f_1^{-1}(x)$ or $f_2^{-1}(x)$ and by $f^{-1}(x) = f_2^{-1}(x)$ we mean that there is only one possibility. If $x \in \delta \in \mathrm{Orb}_+(f)$, $x \in [0,\rho)$ and $x \in \mathrm{Per}(f)$, then for $f^{-1}(x) \in \delta$ we obtain $f^{-1}(x) = f_2^{-1}(x) \in I_1$.

Now consider type 2 mapping. Let $g(x) = f(f(x))$. Dynamical system $(g,[0,s])$ satisfies all conditions for the case 1. Let $\delta \in \mathrm{Orb}_+(f)$, $\delta = \{x_k\}$, $k \in \mathbb{Z}$ and $x_0 \in [0,s)$. It is clear to see that such point x_0 always exists. By the case two subsequences $\{x_{2n}\}$, $n \in \mathbb{Z}$ can be parametrized by I_s. Since f^{-1} is one to one on $[0,s)$, then I_s parametrizes all $\delta \in \mathrm{Orb}_+(f)$. Consider type 1 mapping. It has a cycle of period two which is attracting by Lemma 1. In this case $s = 1$ and β_0 are stable repellent points. Consider mapping $g(x) = f(f(x))$ (see Fig. 3). Define $d = f_1(0)$. It is clear that $(g(x),[0,d])$ is equivalent to type 2 mapping for $i = 1$. Therefore exists interval I_{β_0} that parameterizes orbits $\delta \in \mathrm{Orb}_+(f)$ $\delta = \{x_k\}$ such that $x_k < d$ for all k ($\delta \in P_{\beta_0}$). Define $I_1 = [d, f_2^{-1}(d)]$ and $I_{j+1} = f_2^{-1}(I_j)$, $j \geqslant 1$. Let's prove that any of the intervals I_j can be chosen as I_s. Indeed if $t_1, t_2 \in I_j$ and $t_1 \neq t_2$, then $f_2^{(-n)}(t_1) \neq f_2^{(-n)}(t_2)$ for all $n \geqslant 1$. Therefore different points of the interval correspond to different orbits. If $x_k \in \delta \in \mathrm{Orb}_+(f)$, $x_k > d$, then $x_k \in I_l$ for some l and for all $j \geqslant 1$ there exists $n \in \mathbb{Z}$ such that $x_{k+n} \in I_j$. Hence $P_s \dot\cup P_{\beta_0}$ parameterizes all orbits in $\mathrm{Orb}_+(f)$.

The proof for type 2 and $i = 2$ is absolutely analogous to the proof for type 2 and $i = 1$. ∎

Proposition 1. *Let (f,I) be either type 1 dynamical system and $i = 0$ or type 2 dynamical system and $i = 1$; then it has only one anti-Fock orbit δ, $|\alpha(\delta)| = 1$.*

Proof. For type 1 mapping we can simply write it: $\left\{0,\, \rho_1,\, f_2^{-1}(\rho_1),\, f_2^{(-2)}(\rho_1),\, \dots\right\}$. It is clear that $\lim\limits_{n\to\infty} f_2^{(-n)}(\rho_1) = 1$ exists. Hence $|\alpha(\delta)| = 1$.

For type 2 mapping we consider a sequence $\{0,\, \rho_1,\, g^{-1}(\rho_1),\, g^{(-2)}(\rho_1),\, \dots\}$, where $g(x) = f(f(x))$. Like for type 1 this sequence is a unique anti-Fock orbit for g. Since f^{-1} is one to one on $[0,s]$ we obtain that a unique anti-Fock orbit for g corresponds to a unique anti-Fock orbit for f. ∎

2 Enveloping C^*-algebra

By $C^*(A_f)$ we mean a C^*-algebra obtained from free $*$-algebra $\mathcal{F}(X,X^*)$ generated by X with sub-norm $\|b\| = \sup\limits_{\pi} \|\pi(b)\|$ where supremum is taken over all $\pi \in \mathrm{Rep}(\mathcal{F}(X,X^*))$ such that $\pi(XX^*) = f(\pi(X^*X))$ by standard factorization and completion procedure. The following theorem (see [2]) connects representations of C^*-algebra $C^*(A_f)$ with certain orbits of dynamical system $(f,\mathbb{R}_+)$.

Theorem 2. *Let f be partially monotone continuous map and $(f,\mathbb{R})$ be $\mathcal{F}_{2^m}$ dynamical system. Let $\mathcal{A} = \mathbb{C}^*(A_f)$ be corresponding C^*-algebra.*

1. To every positive non-cyclic orbit $\omega(x_k)_{k\in\mathbb{Z}}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(\mathbb{Z})$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k \in \mathbb{Z}$ and $X = UC$ is a polar decomposition.

2. To positive non-cyclic Fock-orbit $\omega = (x_k)_{k\in\mathbb{N}}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(\mathbb{N})$ given by the formulae: $Ue_0 = 0$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$.

3. To positive non-cyclic anti-Fock-orbit $\omega = (x_{-k})_{k\in\mathbb{N}}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(\mathbb{N})$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$.

4. To cyclic positive orbit $\omega = (x_k)_{k\in\mathbb{N}}$ of length m there corresponds a family of m-dimensional irreducible representation $\pi_{\omega,\phi}$ in Hilbert space $l_2(\{1,\ldots,m\})$ given by the formulae: $Ue_0 = e^{i\phi}e_{m-1}$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k = 1,\ldots,m$; $0 \le \phi \le 2\pi$ and $X = UC$.

This is a complete list of unequivalent irreducible representation of a given $*$-algebra.

As follows from [6] C^*-algebras generated by operators of irreducible representations are either $Z \times_\delta C(\overline{\delta})$, where $\overline{\delta} = \delta \cup \omega(\delta) \cup \alpha(\delta)$ for non-cyclic bilateral orbit or $M_m(\mathcal{T}(C(\mathbb{T})))$, where $\mathcal{T}(C(\mathbb{T}))$ is algebra of the Toeplitz operators for Fock and anti-Fock orbits.

Consider T as a topological space with topology induced from $\mathbb{R}$. Let H be a Hilbert space with orthonormal basis $(e_k)_{k\in\mathbb{Z}}$. Let U be unitary shift operator $Ue_k = e_{k+1}$ for all $k \in \mathbb{Z}$. We know that for any $t \in T$ $\phi(t) = (x_k)_{k\in\mathbb{Z}} \ni t$ futher on we will assume, without loss of generality, that $x_0 = t$. Denote by $C_{\phi(t)}$ diagonal operator $C_{\phi(t)}e_k = x_k e_k$ for all $k \in \mathbb{Z}$. Algebra $C^*(\pi_{\phi(t)})$ is generated by operator $X_{\phi(t)} = U(C_{\phi(t)})^{1/2}$. Denote by $\Psi : C^*(\pi_{\phi(t)}) \to B(H)^T$ the $*$-homomorphism defined on the generator as $\Psi(X)(t) = X_{\phi(t)}$. Further on we will denote by $\pi_{\phi(t)}$ the (irreducible for $\phi(t) \in \mathrm{Orb}_+(f)$ and reducible when $t \in \overline{T} \setminus T$) representation associated with non-cyclic orbit $\phi(t)$ by formulas of the Theorem 2 and by $\pi_{\beta,\psi}$ the finite dimensional representation associated with cycle β and parameter $\psi \in [0, 2\pi]$. In the following theorem we give the description of all irreducible representations of $C^*(A_f)$ in cases 1 and 3 of Lemma 1 as well as fix some notations.

Theorem 3. *Let (f, I) be either type 1 mapping and cycle of period one is attracting or type two mapping and cycle of period two is attracting then*

1. In the first case $C^*(A_f)$ has only one-dimensional irreducible finite dimensional representations parameterized by $\phi, \psi \in [0, 2\pi)$. They are given by the following formulas: $\pi_0(X) = \sqrt{x_0}e^{i\phi}$, $\pi_1(X) = e^{i\psi}$. In the second case $C^*(A_f)$ has only one-dimensional and two-dimensional irreducible finite dimensional representations parameterized by $\phi \in [0, 2\pi)$ they are of the form $\pi_{s,\phi}$ and $\pi_{\beta_1,\phi}$.

2. $C^*(A_f)$ has irreducible Fock representation π_f and one irreducible anti-Fock representation π_{af}. Both of them in case 1 and π_{af} in case 2 generate algebras of Toeplitz operators. In case 2 π_f generate algebra $M_2(\mathcal{T}(C(\mathbb{T})))$, where $\mathcal{T}(C(\mathbb{T}))$ is the algebra of Toeplitz operators.

3. In the first case for each $t \in T = I_1$ there is irreducible infinite-dimensional representations $\pi_{\phi(t)}$ of $C^*(A_f)$. For all $t \in T$ operators of π_t generate isomorphic C^*-algebras. Denote this algebra by $\mathcal{A}$. Algebra $\mathcal{A}$ is a cross-product algebra $C(X) \times \mathbb{Z}$ where X is a closure of any orbit $\phi(t)$. Algebra $\mathcal{A}$ has only one infinite-dimensional representation and two circles of one dimensional representations denote γ_s, γ_1 two arbitrary such representations from different circles. In the second case for each $t \in T = I_s$ there is irreducible infinite-dimensional representations $\pi_{\phi(t)}$ of $C^*(A_f)$. For all $t \in I_s$ operators of $\pi_{\phi(t)}$ generate isomorphic C^*-algebras. Denote this algebra by $\mathcal{B}$. Algebra $\mathcal{B}$ has only one infinite-dimensional representation one circle of one dimensional representations (denote η_s any of them) and one circle of two-dimensional representations (denote η_{β_1} any of them).

4. $*$-algebra $C^*(A_f)$ has no other irreducible representations.

5. For any $a \in C^*(A_f)$ the mapping $\Psi(a)$ is continuous map from T to $B(H)$ where the latter is endowed with norm topology. Moreover, for all $a \in C^*(A_f)$ the following equality holds $\Psi(a)(t_2) = U^*\Psi(a)(t_1)U$, where $\overline{T} = [t_1, t_2]$.

6. C^*-algebras $C^*(\pi_{\phi(t_1)})$ and $C^*(\pi_{\phi(t_2)})$ coincide for any $t_1, t_2 \in T$ as a subalgebras of $B(H)$. We have denoted this algebra by $\mathcal{A}$ for dynamical systems of type 1 and by $\mathcal{B}$ for type 2. Since $U \in \mathcal{A}$ and $U \in \mathcal{B}$ we denote by adU the inner automorphism $a \to U^*aU$, $a \in \mathcal{A}$ or $a \in \mathcal{B}$ as appropriate.

Proof. First two statements of the theorem are direct consequences of Theorems 2, 1. Let us show that for any $t_1, t_2 \in T$ algebras $C^*(\pi_{\phi(t_1)})$ and $C^*(\pi_{\phi(t_2)})$ coincide as a subalgebras of $B(H)$. $C^*(\pi_{\phi(t)})$ is generated by operators U and $C_{\phi(t)}$. Since $\phi(t)$ is not periodic there is point $x \in \phi(t)$ which occurs only finite number of times in the sequence $\phi(t)$, it is easy to see that x is isolated point in $\overline{\phi(t)}$. Hence if we put g to be equal to 1 at x and zero otherwise then g will be continuous function on spec $(C_{\phi(t)})$ and $g(C_{\phi(t)})$ will be a compact non-zero operator in $C^*(\pi_{\phi(t)})$. And since this algebra is prime it contains all compact operators. Hence by compact perturbation $C_{\phi(t)} + K$, where K is compact we can obtain any diagonal operator $C = \operatorname{diag}(c_k)_{k\in\mathbb{Z}}$ such that $\omega(\{c_k\}) = \omega(\phi(t))$ and $\alpha(\{c_k\}) = \alpha(\phi(t))$. Obvious equality $C^*(U, C_{\phi(t)}) = C^*(U, C_{\phi(t)} + K)$ completes the proof of our claim. It is easy to see that, up to isomorphism, $C^*(\pi_{\phi(t)})$ depends only on two integers $|\omega(\phi(t))|$ and $|\alpha(\phi(t))|$.

We proceed now to show that for every $a \in C^*(A_f)$ the map $\Psi(a)$ is continuous. Since X is a generator of $C^*(A_f)$ we need only to prove that $\Psi(X)(t) = U(C_{\phi(t)})^{1/2}$ is continuous in t

$$||\Psi(X)(t) - \Psi(X)(t')|| = \left\| C^{1/2}_{\phi(t)} - C^{1/2}_{\phi(t)} \right\| = \sup_{k\in\mathbb{Z}} \left| x_k^{1/2} - (x'_k)^{1/2} \right|.$$

Hence continuity at t' is equivalent to uniform convergence of $\phi(t)$ to $\phi(t')$ when $t \to t'$. Fix arbitrary $\epsilon > 0$. It can be inferred from the proof of Theorem 1 that if $\phi(t) = (y_s(t))_{s\in\mathbb{Z}}$ then $y_s(t) = g_s(t)$ for $s < 0$ where g_s is a composition of f_1^{-1} and f_2^{-1} and this composition is independent of $t \in T$. Let c_1 be $\alpha(\phi(t))$ and c_2 be $\omega(\phi(t))$ which are independent of $t \in T$. For $\epsilon > 0$ there is integer S such that $y_s(t) \in B_\epsilon(c_1) \cup B_\epsilon(c_2)$ for all $|s| > S$ and t in some neighborhood of t'. Thus we can find $\eta > 0$ such that $\sup\limits_{s:|s|>S} |y_s(t') - y_s(t)| < \epsilon$ for all t' : $|t - t'| < \eta$. Since functions g_s and $f^{(j)}$ are continuous we can choose η small enough for $|g_s(t') - g_s(t)| < \epsilon$ and $|f^{(s)}(t') - f^{(s)}(t)| < \epsilon$ to be true for all s: $|s| \le S$ and $|t - t'| < \eta$, i.e. $\sup\limits_{s:|s|\le S} |y_s(t') - y_s(t)| < \epsilon$. Hence, $\phi(t')$ uniformly converges to $\phi(t)$. Other statements of the theorem are straightforward. ∎

Remark 1. For any $a \in C^*(A_f)$ the map $\Psi(f)$ is a continuous map from $\overline{T}$ to $\mathcal{A}$ for type 1 dynamical systems (or $\mathcal{B}$ for type 2 dynamical systems) such that $\operatorname{ad} U(\Psi(a)(t_2)) = \Psi(a)(t_1)$.

Now we are ready to describe enveloping C^*-algebras. Define operators U_1 and U_2 on the basis as follows $U_1 e_k = e_{k+1}$ for $k < 0$ and $U_1 e_k = 0$ for $k \ge 0$ and $U_2 e_k = e_{k+1}$ for $k > 0$ and $U_1 e_k = 0$ for $k \le 0$. Consider two C^*-subalgebras $\mathcal{G}_1$ and $\mathcal{G}_2$ in $B(H)$ generated by operators U_1 and U_2 correspondingly. Then operator $U_1 + U_2$ generates C^*-subalgebra $\mathcal{G}_1 \oplus \mathcal{G}_2$ in $B(H)$ isomorphic to $\mathcal{T}(C(\mathbb{T})) \oplus \mathcal{T}(C(\mathbb{T}))$. Further on we will use notations of theorem 3 and will regard $\mathcal{T}(C(\mathbb{T})) \oplus \mathcal{T}(C(\mathbb{T}))$ as a concrete algebra in $B(H)$, namely $\mathcal{G}_1 \oplus \mathcal{G}_2$. Let (f, I) be of type 1 with attractive stable point. Let $\mathcal{C}$ denote C^*-algebra of all continuous maps ξ from $\overline{T} = [t_1, t_2]$ to $\mathcal{A}$ such that $\operatorname{ad} U(\xi(t_2)) = \xi(t_1)$.

Theorem 4. *Let (f, I) be of type 1 with attractive stable point. Then $\mathcal{T}(C(\mathbb{T})) \oplus \mathcal{T}(C(\mathbb{T}))$ is a C^*-subalgebra in $\mathcal{A}$. Let us denote by $\mathcal{M}_1$ the C^*-subalgebra in $\mathcal{C}$ comprised of those elements f such that $f(t_1) \in \mathcal{T}(C(\mathbb{T})) \oplus \mathcal{T}(C(\mathbb{T}))$ and $\pi(f(t)) = \pi(f(t'))$ for any one dimensional representation π of $\mathcal{A}$ from the first circle and $\rho(f(t)) = \rho(f(t'))$ for any one dimensional representation ρ of $\mathcal{A}$ from the second circle (see theorem 3) and for all $t, t' \in T$. Then $C^*(A_f)$ is isomorphic to $\mathcal{M}_1$.*

Proof. It is easy to verify that $\pi_{\phi(t_1)}$ is equivalent to the direct sum of Fock and anti-Fock representations. Hence representations $\pi_{\phi(t)}$ where $t \in \overline{T}$ comprise a residual family for $C^*(A_f)$. By Theorem 3 and the remark $C^*(A_f)$ is isomorphic under Gelfand transformation ($\Gamma(a)(\pi) = \pi(a)$, where $\pi \in \mathrm{Rep}(C^*(A_f))$) to a C^*-subalgebra in $\mathcal{C}$. Conditions $\pi(f(t)) = \pi(f(t'))$ and $\rho(f(t)) = \rho(f(t'))$ for all $t, t' \in T$ are easily verified on generator X. Since π and ρ are $*$-homomorphisms these conditions hold for every $a \in C^*(A_f)$. Hence $C^*(A_f)$ is a C^*-subalgebra in $\mathcal{M}_1$. Since it is a massive subalgebra in GCR C^*-algebra $\mathcal{M}_1$ we have $C^*(A_f) = \mathcal{M}_1$ by theorem 11.1.6 [5]. ■

Let $\mathcal{G}_3$ denote the C^*-subalgebra in $B(H)$ generated by operator X_f defined by $X_f e_k = x_k e_{k+1}$ for $k > 0$ and $X_f e_k = 0$ for $k \le 0$ (i.e. X_f is $\pi_f(X)$ if $l_2(\mathbb{N})$ is identified with subspace in $l_2(\mathbb{Z})$). Then operator $U_1 + X_f$ generates $\mathcal{G}_1 \oplus \mathcal{G}_3$ which is isomorphic to $\mathcal{T}(C(\mathbb{T})) \oplus M_2(\mathcal{T}(C(\mathbb{T})))$. Further on we will identify the latter with the concrete C^*-algebra $\mathcal{G}_1 \oplus \mathcal{G}_3$. Let $\mathcal{D}$ denote C^*-algebra of all continuous maps ξ from $\overline{T} = [t_1, t_2]$ to $\mathcal{B}$ such that $\mathrm{ad}\, U(\xi(t_1)) = \xi(t_2)$.

Theorem 5. *Let (f, I) be type two mapping and cycle of period two is attracting. Then $\mathcal{G}_1 \oplus \mathcal{G}_3$ is a subalgebra in $\mathcal{B}$. Let us denote by $\mathcal{M}_2$ the C^*-subalgebra of $\mathcal{D}$ comprised of those elements f such that $f(t_1) \in \mathcal{T}(C(\mathbb{T})) \oplus M_2(\mathcal{T}(C(\mathbb{T})))$ and $\eta(f(t)) = \eta(f(t'))$ for any one dimensional representation η of $\mathcal{B}$ and $\zeta(f(t)) = \zeta(f(t'))$ for any two dimensional representation ζ of $\mathcal{B}$ and for all $t, t' \in T$. Then $C^*(A_f)$ is isomorphic to $\mathcal{M}_2$.*

The proof is analogous to that of the previous theorem.

Corollary 1. *For $\mathcal{F}_{2^n}$ unimodal dynamical systems with zero Schwarzian and attractive cycle of length one or two isomorphism class of associated C^*-algebra depends only on the type of the system (whether it 1 or 2).*

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