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On the Heisenberg–Lie Algebra and Some Non-Hermitian Operators in Oscillatorlike Developments

Jules BECKERS and Nathalie DEBERGH

Theoretical Physics, Institute of Physics (B5), University of Liège, B-4000 Liège 1, Belgium
E-mail: *Jules.Beckers@ulg.ac.be, Nathalie.Debergh@ulg.ac.be*

After a few generalities we compare fundamental quantum mechanics applied to the harmonic oscillator with unusual oscillatorlike developments dealing with non-hermitian operators, this latter aspect exploiting in particular the property of subnormality. In this last context we can also restore the hermiticity of the Hamiltonian operator and discover a nice property of the new scalar product. General constructions of oscillatorlike Hamiltonians are also considered and new ideas connected with Heisenberg relations are presented.

1 Introduction

In every treatise on Quantum Mechanics [1] we learn that physical observables are represented by linear and self-adjoint operators acting on states belonging to Hilbert spaces characterized by a well defined scalar product. It is the case for important observables such as position, momentum and energy in particular, the last one being in correspondence with the (quantum) Hamiltonian operator (ensuring by its self-adjointness to have a real spectrum). The operators generate between themselves a Lie algebra which in the context of position and momentum is called the Heisenberg algebra

$$[x, p] = iI, \qquad [x, x] = [p, p] = 0 \qquad (\hbar = 1).$$

When the one-dimensional harmonic oscillator is described, this algebra can be put on the form

$$[a, a^\dagger] = I, \qquad [a, a] = [a^\dagger, a^\dagger] = 0 \qquad (\omega = 1), \tag{1}$$

where

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right), \qquad (a^\dagger)^\dagger = a,$$

a being known as the annihilation operator and $a^\dagger$ as the creation one, acting on Fock states $\{|n\rangle, n = 0, 1, 2, \ldots\}$. Within such developments, the Hamiltonian

$$H_{\text{H.O.}} = \frac{1}{2}\{a, a^\dagger\} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2$$

has the *real* spectrum

$$E_n = n + \frac{1}{2} \tag{2}$$

and the set of eigenfunctions

$$h_n(x) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{2^n n!}} e^{-\frac{x^2}{2}} H_n(x),$$

where H_n are the well known (classical) Hermite polynomials. Let us also recall that we have the so-called intertwining relations

$$[H_{\mathrm{H.O.}}, a] = -a, \qquad [H_{\mathrm{H.O.}}, a^\dagger] = a^\dagger \tag{3}$$

characteristic of these developments. It is important to notice here that all these elements are ad-hoc ones for getting interesting constructions of the famous coherent states [2] but not for visiting squeezed states [3].

Let us now enter a recent property called "subnormality of unbounded operators" [4] and let us mention its definition: "a densely defined Hilbert space operator S is said to be subnormal if there is a normal operator N (acting possibly in a larger space) such that S is included in N". Mathematicians have shown that "the best behaving unbounded subnormal operator is the famous creation operator $a^\dagger$ of the harmonic oscillator acting in $L^2(\mathbb{R})$". It is equivalent to the one defined by

$$a^\dagger_\lambda = a^\dagger + \lambda I, \qquad \lambda \in \mathbb{R},$$

an interesting way to introduce a new supplementary parameter in oscillatorlike developments.

By noticing that the previous commutation relations (1) and (3) are unchanged by the substitution $a^\dagger \to a^\dagger_\lambda$, we point out that the new Hamiltonian H_λ becomes

$$H_\lambda = H_{\mathrm{H.O.}} + \lambda a.$$

It is no more self-adjoint (notice that $(a^\dagger_\lambda)^\dagger \neq a$) *but* still has a real spectrum (2) and eigenfunctions depending now on λ. Indeed we get [5] a Fock basis characterized by

$$|n\rangle_\lambda \equiv \psi_n(\lambda, x) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{2^n n!}} \frac{1}{\sqrt{L_n^{(0)}(-\lambda^2)}} e^{-\frac{x^2}{2}} H_n\left(x + \frac{\lambda}{\sqrt{2}}\right)$$

which has the new interesting property to lead here to meaningful squeezed states [5]. We have thus deformed the states but without deforming the Lie algebra.

The lost of the self-adjointness of H_λ is evident with respect to the original scalar product but we have noticed that this well accepted property can be restored if we modify [6] the scalar product by asking that

$$\left(a^\dagger_\lambda \psi_n(\lambda, x), \psi_m(\lambda, x)\right) = (\psi_n(\lambda, x), a\psi_m(\lambda, x)) = \delta_{nm}.$$

Such a property evidently corresponds to $(a^\dagger_\lambda)^\dagger = a$ in the new Hilbert space characterized by a measure depending on λ. It is not difficult to prove [6] that the new measure is given in a unique way by

$$\rho(\lambda, x)dx = \exp\left[-\sqrt{2}\lambda x - \frac{\lambda^2}{2}\right] dx,$$

where we recognize the generating function of the Hermite polynomials.

Unhappily this context only gives new families of coherent states but no information on squeezed states.

In order to include these $\lambda \neq 0$ and $\lambda = 0$ contexts, we have proposed [7] a general construction of oscillatorlike Hamiltonians by maintaining $(a^\dagger)^\dagger = a$ but permitting $H^\dagger = H$ or $H^\dagger \neq H$. Let us now construct the following operators

$$b = (1 + c_1)a + c_2 a^\dagger + c_3 \qquad \text{and} \qquad b^+ = c_4 a + (1 + c_5)a^\dagger + c_6$$

and require that

$$H = \frac{1}{2}\{b, b^+\}, \qquad [b, b^+] = 1, \qquad [H, b] = -b, \qquad [H, b^+] = b^+.$$

We point out that the usual harmonic oscillator corresponds to all the c_i's equal to zero and our deformed context to all the c_i's equal to zero except the sixth one ($c_6 = \lambda$). Moreover by asking for Schrödinger Hamiltonians of the type

$$H = A\frac{d^2}{dx^2} + (Bx + C)\frac{d}{dx} + Dx^2 + Ex + F,$$

we can easily get A, B, C, D, E, F as functions of the c_i's parameters and see that if $A = -D = -\frac{1}{2}$, $B = C = E = F = 0$, we recover the harmonic oscillator and if $A = -D = -\frac{1}{2}$, $C = E = \frac{\lambda}{\sqrt{2}}$, $B = F = 0$, we recover our λ-context. Moreover we notice that $H^\dagger = H$ iff $B = C = 0$ while $H^\dagger \neq H$ in the other cases. All these developments are subtended by new Fock spaces $\{|n\rangle_c, n = 0, 1, 2, \ldots\}$ characterized by square integrable eigenfunctions (when $A < 0$ and $B < 1$) associated to real eigenvalues $E_n = n + \frac{1}{2}$ and the action of the operators given by

$$\begin{aligned}
&b|n\rangle_c = \frac{n}{\sqrt{-A}}\frac{N_n}{N_{n-1}}(1 + c_1 - c_2)|n-1\rangle_c, \\
&b^+|n\rangle_c = \frac{1}{2\sqrt{-A}}\frac{N_n}{N_{n+1}}(1 + c_5 - c_4)|n+1\rangle_c, \\
&b^+b|n\rangle_c = n|n\rangle_c, \qquad bb^+|n\rangle_c = (n+1)|n\rangle_c.
\end{aligned}$$

These results give once more new results [7] in quantum optics through coherence and squeezing developments.

Let us end this communication by two comments dealing once again with non-hermitian operators and some surprising results.

The *first* comment is connected to the choice of unusual annihilation and creation operators illustrating the $(a^\dagger)^\dagger \neq a$ context. Following Ushveridze [8], we can choose

$$a = \frac{1}{\alpha'(x)}\left(\frac{d}{dx} + \beta(x)\right), \qquad a^\dagger = \alpha(x) \tag{4}$$

which ensure the Heisenberg algebra (2) and lead to

$$H = a^\dagger a + \frac{1}{2} \to E_n = n + \frac{1}{2}, \qquad f_n(x) = N_n \exp\left[-\int \beta(x)dx\right] A^n(x). \tag{5}$$

With the Ushveridze specific values

$$a = \frac{d}{dx} + cx^3, \qquad a^\dagger = x, \qquad c > 0, \tag{6}$$

one finds square integrable eigenfunctions which are normalizable but not orthogonal. Our surprise was that the construction of the orthogonal ones by the Schmidt procedure leads to eigenfunctions containing the so-called Freud polynomials [9] permitting once again squeezing developments [10] in this special matter.

The *second* comment is on the position and momentum operators discussed from equations (4)–(6). In fact we immediately have new operators given by

$$X = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right) = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + cx^3 + x\right) \tag{7}$$

and

$$P = \frac{i}{\sqrt{2}}\left(a - a^{\dagger}\right) = \frac{i}{\sqrt{2}}\left(-\frac{d}{dx} - cx^3 + x\right) \tag{8}$$

which evidently are such that $X^{\dagger} \neq X$ and $P^{\dagger} \neq P$. Our proposal [11] is to rewrite (7) and (8) in the following forms

$$X = \operatorname{Re} X + i \operatorname{Im} X = \frac{1}{\sqrt{2}}\left(x + cx^3\right) + i\left(-\frac{i}{\sqrt{2}}\frac{d}{dx}\right)$$

and

$$P = \operatorname{Re} P + i \operatorname{Im} P = \left(-\frac{i}{\sqrt{2}}\frac{d}{dx}\right) + \frac{i}{\sqrt{2}}\left(x - cx^3\right)$$

putting in evidence four "new" operators which are self-adjoint while X and P are not. Moreover we have

$$[X, P] = iI = [\operatorname{Re} X, \operatorname{Re} P] - [\operatorname{Im} X, \operatorname{Im} P],$$

a trivial generalization of the usual Heisenberg characteristics. If we come back on $X = x$ and $P = -i\frac{d}{dx}$, we get

$$\operatorname{Im} X = 0 = \operatorname{Im} P$$

and the usual commutation relation takes place in correspondence with

$$\Delta x \Delta p \geq \frac{1}{2}.$$

Different questions are now open and have to be discussed in the future.

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