Proceedings of Institute of Mathematics of NAS of Ukraine 2002, Vol. 43, Part 1, 86–92

Group Classification of Nonlinear Partial Differential Equations: a New Approach to Resolving the Problem

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We describe a systematic procedure for classifying partial differential equations which are invariant with respect to low-dimensional Lie algerbas. This procedure is a synthesis of the infinitesimal Lie method, the technique of equivalence transformations and the theory of classification of abstract low-dimensional Lie algebras. By way of illustration, we consider three examples of group classification of partial differential equations in new approach.

1 Introduction

This article is based on two talks (one given by each of the authors) at the Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–14 July, 2001, Kyiv, Ukraine). More details can be found in [9] and [16], and a short description is given in [17].

The analysis and classification of differential equations using group theory goes back to Sophus Lie. The first systematic investigation of the problem of group classification was done by L.V. Ovsiannikov [1] in 1959 for nonlinear heat equation

$$u_t = [f(u)u_x]_x,$$

where $f(u)$ is an arbitrary nonlinearity. His approach is based on the concept of the equivalence group, which is the Lie transformation group (acting in the space whose local coordinates are independent variables, the functions and their derivatives) preserving the class of particular differential equations under study. It is possible to modify Lie's algorithm in order to make it applicable for the computation of this group (see, e.g., [2]). Having obtained the equivalence group one constructs the optimal system of subgroups of the equivalence group. The last step uses Lie's algorithm for obtaining specific partial differential equations that (a) belong to the class under study, and (b) are invariant with respect to these subgroups.

This approach has been applied to a number of equations of mathematical physics. Here we mention just a few of the papers in which the group classification of nonlinear heat equations has been studied:

Akhatov, Gazizov, Ibragimov (1987, [3])

$$u_t = G(u_x)u_{xx};$$

Dorodnitsyn (1982, [4])

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + g(u);$$

Oron, Rosenau (1986, [5]), Edwards (1994, [6])

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + f(u)u_x;$$

Cherniha and Serov (1998, [7])

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + f(u)u_x + g(u);$$

Gandarias (1996, [8])

$$u_t = u^n u_{xx} + nu^{n-1}u_x^2 + g(x)u^m u_x + f(x)u^s.$$

However, the possibility of implementing Ovsiannikov's approach in its full generality presupposes that we are able to construct the optimal system of subgroups of the equivalence group. However, even for the case when the equivalence group is finite-parameter, there arise major algebraic difficulties, since the classification problem for all finite-parameter Lie groups has not yet been solved (to say nothing about infinite-parameter Lie groups, where this problem is completely open). Consequently, there is an evident need for Ovsiannikov's approach to be modified so as to be applicable to the case of infinite-parameter equivalence groups.

Here we turn out attention to a new approach, proposed by R. Zhdanov and V. Lahno in [9], that enables us to solve efficiently the symmetry classification problem for partial differential equations even for the case of infinite-dimensional equivalence groups. It is based mainly on the following facts:

- If the partial differential equation possesses non-trivial symmetry, then it is invariant under some finite-dimensional Lie algebra of differential operators. In the event that the maximal algebra of invariance is infinite-dimensional, then it contains, as a rule, some finite-dimensional Lie algebra.
- If there are local non-singular changes of variables which transform a given differential equation into another, then the finite-dimensional Lie algebras of invariance of these equations are isomorphic, and in the group-theoretic analysis of differential equations such equations are considered to be equivalent.
- Abstract Lie algebras of up to six dimensions have already been classified [10, 11, 12, 13].

What we have in [9] is a preliminary classification of inequivalent realizations of low-dimensional Lie algebras within some specific class of first-order linear differential operators. This class is determined by the structure of the equation under study. Its elements form a representation space for realizations of Lie algebras of symmetry groups admitted by the equations belonging to the class of partial differential equations under study. A natural equivalence relation is introduced on the set of all possible realizations. Namely, two realizations are called equivalent if they are transformed into each other by the action of the equivalence group. In other words, solving the problem of symmetry classification of partial differential equations having some prescribed form, is equivalent to constructing a representation theory of Lie transformation groups (or Lie algebras of first-order partial differential operators) realized as symmetry groups (algebras) of the equations in question.

2 Description of the method

The new approach to the classification of partial differential equations is a synthesis of Lie's infinitesimal method, the use of equivalence transformations and the theory of classification of abstract finite-dimensional Lie algebras. It provides a constructive solution of the problem of the group classification of partial differential equations possessing arbitrary elements and admitting *non-trivial finite-dimensional* invariance algebras.

The group classification in the approach described here is implementation of the following algorithm:

I. The first step involves finding the form of the infinitesimal operators which generate the symmetry group of the equation under consideration, and the construction of the equivalence group of this equation. To find the form of the infinitesimal operators one uses the usual Lie algorithm. In doing this we obtain a system of linear partial differential equations of first order which connect the coefficients of the infinitesimal operators with the arbitrary term of the equation. In the following we call this system *the characterizing system of the equation*. In order to construct the equivalence group $\mathcal{E}$ of the equation one can use the infinitesimal method as well as the direct method.

II. In the second step one carries out the group classification of those equations of the given form which admit finite-dimensional Lie algebras of invariance.

   For this, one carries out a step-by-step classification of finite-dimensional Lie algebras within the specified class of infinitesimal operators, up to equivalence under transformations of the group $\mathcal{E}$. In doing this, one has to check that each algebra obtained in this way can be an invariance algebra of the equation at hand before proceeding from the realization of Lie algebras of lower dimension to the realization of Lie algebras of higher dimension. This eliminates superfluous realizations of Lie algebras. Also, those realizations of Lie algebras which are invariance algebras of the equation will, as their dimension increases, correspond to greater fixing of the arbitrary term.

   This procedure is continued until the arbitrary term in the equation is completely determined or until it is no longer possible to extend the realization of Lie algebras beyond a given dimension within the specified class of infinitesimal operators.

III. The third step is then to exploit the characterizing system or the infinitesimal method of Lie in order to find, for each of the particular choices of the arbitrary term, the maximal invariance algebra of the equation under consideration. Moreover, the equivalence of the equations obtained in this manner is determined. We note that, in as much as equivalent equations have isomorphic invariance algebras, we may test the realizations of the invariance algebras for equivalence rather than test the equations themselves.

3 Examples of the group classification

Here we give some examples illustrating how the method works.

Example 1 ([14]). Group classification of

$$u_{tx} + A(t,x)u_t + B(t,x)u_x + C(t,x)u = 0. \tag{1}$$

Ovsiannikov [15] gave a group classification of (1), using Laplace invariants

$$h = A_t + AB - C, \qquad k = B_x + AB - C.$$

His results can be formulated as follows:

Theorem 1. *Equation (1) admits a Lie symmetry algebra of dimension greater than 1 if and only the functions p, q given by*

$$p = \frac{k}{h}, \qquad q = \frac{1}{h}\partial_x\partial_y(\ln h)$$

are constant. In this case, equation (1) is equivalent either to the Euler–Poisson equation

$$u_{tx} - \frac{2u_t}{q(t+x)} - \frac{2pu_x}{q(t+x)} + \frac{4pu}{q^2(t+x)^2} = 0$$

when $q \neq 0$*, or to the equation*

$$u_{tx} + tu_t + pxu_x + ptxu = 0$$

when $q = 0$*.*

We have carried out the group classification of equation (1) using our method.

First, we find (by standard methods) that the infinitesimal generator of symmetries is given by

$$X = f(t)\partial_t + q(x)\partial_x + h(t,x)u\partial_u,$$

where the functions f, g, h satisfy

$$\begin{aligned}
&h_t + B\dot{f} + fB_t + gB_x = 0,\\
&h_x + Ag' + gA_x + fA_t = 0,\\
&h_{tx} + C\dot{f} + fC_t + Cg' + gC_x + Ah_t + Bh_x = 0
\end{aligned} \qquad (2)$$

(we omit the trivial symmetry $X = \omega(t,x)\partial_u$, where ω is an arbitrary solution of (1)).

A direct analysis of (2) is not possible. The **equivalence group** of (1) is given by transformations of the two following types:

$$\begin{aligned}
&(a)\quad r = \alpha(t), \quad \xi = \beta(x), \quad v = \theta(t,x)u + \rho(t,x);\\
&(b)\quad r = \alpha(x), \quad \xi = \beta(t), \quad v = \theta(t,x)u + \rho(t,x),
\end{aligned}$$

where α, β are arbitrary smooth functions and θ, ρ satisfy

$$\theta_t\rho_x + \rho_t\theta_x - \theta\rho_{tx} + \rho\theta_{tx} - 2\frac{\rho}{\theta}\theta_t\theta_x + A[\theta_t\rho - \theta\rho_t] + B[\theta_x\rho - \theta\rho_x] - C\theta\rho = 0.$$

We note that equation (1) is invariant under the operator $u\partial_u$ and that $[X, u\partial_u] = 0$. So, X and $u\partial_u$ form a two-dimensional Lie algebra. There are only two canonical forms for a two-dimensional Lie algebra

$$\begin{aligned}
&A_{21} = \langle e_1, e_2\rangle \quad \text{with} \quad [e_1, e_2] = 0,\\
&A_{22} = \langle e_1, e_2\rangle \quad \text{with} \quad [e_1, e_2] = e_2,
\end{aligned}$$

and we clearly see that only A_{21} is suitable for our purposes.

We now need to find a canonical form for the operator X. We have the following result:

Proposition 1. *Let* A_{21} *be the invariance algebra of equation (1). There are two inequivalent canonical realizations of* $A_{21} = \langle u\partial_u, X\rangle$*:*

$$A^1_{21} = \langle u\partial_u, \partial_t\rangle, \qquad A^2_{21} = \langle u\partial_u, \partial_t + \partial_x\rangle,$$

and the corresponding canonical forms for equation (1) are

$$A^1_{21}: \ u_{tx} + B(x)u_x + u = 0, \qquad (3)$$

$$A^2_{21}: \ u_{tx} + B(z)u_x + C(z)u = 0 \qquad (4)$$

with $z = t - x$*.*

The system (2) for equation (3) then becomes

$$h_t + B\dot{f} + gB_x = 0, \qquad h_x = 0, \qquad \dot{f} + g' = 0, \tag{5}$$

where $B = B(x)$.

We easily integrate (5) and we find $B = mx$, where $m = \text{const} \neq 0$, and equation (3) takes on the form

$$u_{tx} + mxu_x + u = 0.$$

The invariance algebra of this equation is

$$\langle u\partial_u, \partial_t, t\partial_t - x\partial_x, \partial_x - mtu\partial_u \rangle.$$

For equation (4) we find (using the same procedure) that the corresponding canonical form for equation (1) is

$$u_{tx} + \frac{m}{z}u_x + \frac{k}{z^2}u = 0,$$

where m, k are constants with $k \neq 0$, and $z = t - x$.

The invariance algebra of this equation is

$$\langle u\partial_u, \partial_t + \partial_x, t\partial_t + x\partial_x + \frac{1}{2}mu\partial_u, t^2\partial_t + x^2\partial_x + mtu\partial_u \rangle.$$

These results are equivalent to the ones obtained by Ovsiannikov.

Example 2 ([9]). Group classification of nonlinear equation of the form

$$u_t = u_{xx} + F(t, x, u, u_x). \tag{6}$$

First, we find that the infinitesimal generator of symmetries is given by

$$X = 2a(t)\partial_t + (\dot{a}(t)x + b(t))\partial_x + f(t, x, u)\partial_u,$$

where functions a, b, f, F fulfil relation

$$\begin{aligned} f_t = u_x(\ddot{a}x + \dot{b}) + (f_u - 2\dot{a})F = f_{xx} + 2u_x f_{xu} + u_x^2 f_{uu} + 2aF_t \\ + (\dot{a}x + b)F_x + fF_u + f_x F_{u_x} + u_x(f_u - \dot{a})F_{u_x}. \end{aligned} \tag{7}$$

A direct analysis of (7) is not possible.

Using our approach we have established that there are three classes of equations (6) invariant with respect to one-parameter groups, seven classes of equations (6) invariant with respect to two-parameter groups, 28 classes of equations (6) invariant with respect to three-parameter groups and 11 classes of equation (6) invariant with respect to four-parameter groups.

Here we present all representatives of 11 classes of equations (6) invariant with respect to four-parameter groups only:

1. $u_t = u_{xx} + \dfrac{\lambda\epsilon u_x}{4\sqrt{|t|}} \ln|tu_x^2| + \dfrac{\beta u_x}{\sqrt{|t|}},$

   $\epsilon = 1$ for $t > 0$, $\epsilon = -1$ for $t < 0$, $\beta \in \mathbb{R}$, $\lambda \neq 0$;
2. $u_t = u_{xx} - \lambda u_x(x + \ln|u_x|)$, $\lambda \neq 0$;
3. $u_t = u_{xx} + \lambda \exp(-u_x)$, $\lambda \neq 0$;
4. $u_t = u_{xx} + 2\ln|u_x|$;
5. $u_t = u_{xx} - u_x \ln|u_x| + \lambda u_x$, $\lambda \in \mathbb{R}$;

6. $u_t = u_{xx} + \lambda u_x^{\frac{2k-2}{2k-1}}, \quad \lambda \neq 0, \quad k = 0, \dfrac{1}{2}, 1;$
7. $u_t = u_{xx} + \dfrac{1}{4t} u_x^2;$
8. $u_t = u_{xx} - u u_{xx} + \lambda |u_x|^{\frac{3}{2}}, \quad \lambda \neq 0;$
9. $u_t = u_{xx} + \lambda^{-1} x + m\sqrt{|u_x|}, \quad \lambda > 0, \quad m \neq 0;$
10. $u_t = u_{xx} - \dfrac{1}{4}\lambda\epsilon(1-q)|t|^{-\frac{1}{2}(1+q)} u_x^2,$
$\lambda \neq 0, \quad |q| \neq 1, \quad \epsilon = 1 \ \text{ for } \ t > 0, \quad \epsilon = -1 \ \text{ for } \ t < 0;$
11. $u_t = u_{xx} - \dfrac{1}{2}\dot{\alpha} u_x^2 (\lambda - \alpha)\left(1 + \alpha^2\right)^{-1}, \quad \lambda \in \mathbb{R}.$

Note that case 8) with $\lambda = 0$ gives rise to the Burgers equation

$$u_t = u_{xx} - u u_x,$$

which is invariant under a five-parameter group.

Example 3 ([16]). Group classification of nonlinear equations of the form

$$u_t = F(t, x, u, u_x) u_{xx} + G(t, x, u, u_x). \tag{8}$$

In [16] we find that the infinitesimal generator of symetries is given by

$$X = a(t)\partial_t + b(t, x, u)\partial_x + c(t, x, u)\partial_u,$$

where a, b, c are real-valued functions that satisfy the system of particular differential equations

$$\begin{aligned}
&(2b_x + 2u_x b u - \dot{a})F = aF_t + bF_x + cF_u + \left(c_x + u_x c_u - u_x b_x - u_x^2 b u\right) F_{u_x},\\
&c_t - u_x b_t + (c_u - \dot{a} - u_x b_u)G + (u_x b_{xx} - c_{xx} - 2u_x c_{xu} - u_x^2 c_{uu}\\
&\qquad + 2u_x^2 b_{xu} + u_x^3 b_{uu})F = aG_t + bG_x + cG_u + \left(c_x + u_x c_u - u_x b_x - u_x^2 b_u\right) G_{u_x}.
\end{aligned} \tag{9}$$

A direct analysis of system (9) is also not possible.

The principal result [16] of group classification of equations (8) is the following:

Proposition 2. *Equation (8) admits a Lie symmetry algebra of dimension greater than 4 if it is equivalent to one of the following equations:*

1. $u_t = u^{-4} u_{xx} - 2u^{-5} u_x^2;$
2. $u_t = u_{xx} + x^{-1} u u_x - x^{-2} u^2 - 2x^{-2} u;$
3. $u_t = \exp(u_x) u_{xx};$
4. $u_t = u_x^n u_{xx}, \quad n \geq -1, \quad n \neq 0;$
5. $u_t = \exp(n \arctan u_x)\left(1 + u_x^2\right)^{-1} u_{xx}, \quad n \geq 0.$

These equations are invariant under five-dimensional Lie algebras.

Note that equation 1) is equivalent to the equation obtained by Ovsiannikov [1], equation 2) is equivalent to Burgers equation, and equations 3)–5) was obtained by Akhatov, Gazizov and Ibragimov [3].

Acknowledgements

The authors gratefully acknowledge Prof. R.Z. Zhdanov's collaboration in this research.

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