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Invariant Differential Operators and Representations with Spherical Orbits

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It is known that the algebra $\mathcal{D}(V)^G$ of G-invariant differential operators corresponding to a G-module V of a complex reductive group G is commutative if and only if V is a spherical G-module. In the present work we study the structure of $\mathcal{D}(V)^G$ for G-modules with spherical orbits. It is proved that the centralizer $\mathcal{Z}(V)^G$ of the subalgebra $k[V]^G$ in $\mathcal{D}(V)^G$ is commutative. Also a characterization of actions with spherical orbits in terms of the reduced action is obtained.

1 Multiplicity-free representations and spherical varieties

Let G be a connected reductive algebraic group defined over an algebraically closed field of zero characteristic, and (ρ, V) be a finite-dimensional representation of the group G. The induced representation of G on the algebra of polynomials $k[V]$ is given by the formula $(g * f)(v) := f\left(\rho\left(g^{-1}\right)v\right)$ for any $g \in G$, $f \in k[V]$, $v \in V$. It is well known that $k[V]$ as a G-module has the isotypic decomposition

$$k[V] = \oplus_{\lambda \in \Xi_+(G)} k[V]_\lambda,$$

where $\Xi_+(G)$ is the semigroup of dominant weights of G and $k[X]_\lambda$ is the sum of all irreducible G-submodules in $k[V]$ with the highest weight λ.

Definition 1. A representation (ρ, V) is called *multiplicity-free* if for any $\lambda \in \Xi_+(G)$ such that $k[V]_\lambda \neq 0$ the G-module $k[V]_\lambda$ is irreducible. We say in this case that the G-module $k[V]$ is *multiplicity-free*.

A complete list of multiplicity-free irreducible linear actions of connected reductive groups obtained by V. Kac [1, Theorem 3] is as follows:

(1) SL_n, Sp_n, $SO_n \otimes k^*$, S^2GL_n, Λ^2SL_n (for n odd), Λ^2GL_n (for n even), $SL_m \otimes SL_n$ (for $m \neq n$), $GL_n \otimes SL_n$, $GL_2 \otimes Sp_n$, $GL_3 \otimes Sp_3$, $GL_4 \otimes Sp_4$, $SL_n \otimes Sp_4$ (for $n > 4$), $Spin_7 \otimes k^*$, $Spin_9 \otimes k^*$, $Spin_{10}$, $G_2 \otimes k^*$, $E_6 \otimes k^*$.

(2) $G \otimes k^*$ for all semisimple groups G from list (1).

Here k^* is the multiplicative group of the field k considered as a one-dimensional algebraic group. The linear group Λ^2SL_n is the image of SL_n under the representation in the second exterior power of the tautological representation, and S^2SL_n is the same thing with respect to the second symmetric power.

A classification of reducible multiplicity-free representations was obtained independently by C. Benson and G. Ratkliff [2], and by A. Leahy [3].

Multiplicity-free representations form a very restricted class of representations. Nevertheless they are very important due to Roger Howe's philosophy that every "nice" result in the invariant theory of particular representations can be traced back to a multiplicity-free representation. For example, all of Weyl's first and second fundamental theorems can be explained by some multiplicity freeness results. Some other examples we shall discuss below.

Let B be a Borel subgroup of G.

Definition 2. A normal algebraic variety X with regular G-action (and the action $G : X$ itself) is said to be *spherical* if there exists a point $x \in X$ such that the orbit Bx is open in X.

Denote by $k(X)$ the field of rational functions on a variety X and by $k(X)^L$ (resp. $k[X]^L$) the subfield (resp. the subalgebra) of L-fixed elements for any subgroup $L \subset G$. By Rosenlicht's theorem [4, 2.3], the G-variety X is spherical if and only if $k(X)^B = k$.

Theorem 1 ([5]). *Suppose that X is a normal affine variety. Then an action $G : X$ is spherical if and only if the G-module $k[X]$ is multiplicity-free.*

In particular, multiplicity-free representations are in the natural one-to-one correspondence with spherical linear actions.

For more information on interconnections between spherical actions and representation theory, symplectic geometry, classical mechanics and so on, see the recent survey [6].

2 Representations with spherical orbits

In this section we consider a generalization of the notion of spherical action.

Definition 3. Let X be an irreducible algebraic variety. An action $G : X$ is called *an action with spherical orbits* if there exists an open susbet $X_0 \subset X$ such that for any $x \in X_0$ the orbit Gx is a spherical G-variety.

Below we list some basic facts about actions with spherical orbits.

(1) Any spherical actions is an action with sperical orbits.

(2) Any trivial G-actions is an action with spherical orbits.

(3) Rosenlicht's theorem implies that an action $G : X$ is an action with spherical orbits if and only if $k(X)^G = k(X)^B$.

(4) It is shown in [7, Corollary 1] that for an action with spherical orbits any G-orbit is spherical.

(5) Let $G_1 : X_1$ and $G_2 : X_2$ be actions with spherical orbits. Then the action $(G_1 \times G_2) : (X_1 \times X_2)$ is an action with spherical orbits.

Now we consider a fragment of a classification of representations with spherical orbits [8].

Definition 4. A G-module V is *indecomposable* if there exist no proper decompositions $G = G_1 \times G_2$ and $V = V_1 \oplus V_2$ such that $(g_1, g_2) * (v_1, v_2) = (g_1 v_1, g_2 v_2)$ for any $g = (g_1, g_2) \in G$ and any $v = (v_1, v_2) \in V$.

By property (5), it is sufficient to classify indecomposable representations with spherical orbits. In Tables 1 and 2 all indecomposable representations with spherical orbits (but non-spherical!) for connected semisimple groups are indicated. Table 1 contains representations with a one-dimensional quotient (i.e., $k[V]^G = k[q_1]$), and Table 2 contains representations with a two-dimensional quotient (i.e., $k[V]^G = k[q_1, q_2]$). (Here q_i are basic invariants.) There is no indecomposable representations with spherical orbits and a higher-dimensional quotient, for more details see [8].

Comments to the Tables. In the column "weights" the highest weights of the G-module are indicated. For the group $G_1 \times G_2$ the weight $\phi \otimes \psi$ corresponds to the tensor product of simple G_1- and G_2-modules with highest weights ϕ and ψ respectively. The symbol + denotes a direct sum of modules. If G is the product of several simple groups, then their fundamental weights are denoted successively by letters ϕ_i, ψ_i and τ_i.

Table 1.

	G	weights	$\dim V$
0	$\{e\}$	0	1
1	$\Lambda^2 SL_{2n}$	ϕ_2	$2n^2 - n$
2	$S^2 SL_n$	$2\phi_1$	$n(n+1)/2$
3	SO_n, $n > 2$	ϕ_1	n
4	$Spin_7$	ϕ_3	8
5	$Spin_9$	ϕ_4	16
6	G_2	ϕ_1	7
7	E_6	ϕ_1	27
8	SL_n, $n > 2$	$\phi_1 + \phi_{n-1}$	$2n$
9	SL_{2n+1}	$\phi_1 + \phi_2$	$(2n+1)(n+1)$
10	SL_{2n}	$\phi_1 + \phi_2$ $\phi_1 + \phi_{2n-2}$	$n(2n+1)$
11	$SL_n \times SL_n$	$\phi_1 \otimes \phi_1$	n^2
12	$SL_2 \times Sp_{2n}$	$\phi_1 \otimes \phi_1$	$4n$
13	$SL_4 \times Sp_4$	$\phi_1 \otimes \phi_1$	16
14	$SL_n \times SL_2 \times Sp_{2m}$, $n > 2$, $m \geq 1$	$\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$	$2(n+2m)$

Table 2.

	G	weights	$\dim V$
1	SO_8	$\phi_1 + \phi_3$	16
2	$Sp_{2n} \times SL_2 \times Sp_{2m}$, $n, m \geq 1$	$\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$	$4(m+n)$

3 Invariant differential operators

Let X be an affine variety, and set $A = k[X]$. We define the algebra of (algebraic) differential operators on A and X as follows: If $P \in \mathrm{End}_k(A)$ and $a \in A$, then $[P, a]$ denotes the usual commutator: $[P, a](b) = P(ab) - a(P(b))$, $b \in A$. Define $D^n(A) = 0$ for $n < 0$, and for $n \geq 0$ inductively define:

$$D^n(A) = \{P \in \mathrm{End}_k(A) \mid [P, a] \in D^{n-1}(A) \text{ for all } a \in A\}.$$

Clearly, $D^0(A) \cong A$ acting on itself by multiplication. Note that $D^n(A) \subset D^{n+1}(A)$ for all n, and we define $D(A) := \cup_n D^n(A)$. Now we set $\mathcal{D}^n(X) := D^n(A)$, and similarly for $\mathcal{D}(X)$. We call $\mathcal{D}(X)$ *the algebra of differential operators on* X.

Suppose that $X = k^n$, so that $A = k[x_1, \ldots, x_k]$. Then $\mathcal{D}(X)$ is the *kth Weyl algebra* W_k, i.e., the noncommutative algebra $k < x_1, \ldots, x_k, \partial_1, \ldots, \partial_k >$ generated by the x_i and the $\partial_j := \partial/\partial x_j$ with there usual commutation relations.

Now let X be an affine G-variety, where G is complex reductive. The group G acts rationally on $k[X]$ and $\mathcal{D}(X)$ [9, § 3]. Denote by $\mathcal{D}(X)^G$ the algebra of G-invariant differential operators.

We shall need the following well-known result.

Proposition 1. *If X is a spherical G-variety, then the algebra $\mathcal{D}(X)^G$ is commutative.*

Proof. By Schur's Lemma, any endomorphism T of $k[X]$ which commutes with G must preserve each isotypic component $k[V]_\lambda$. Further, the restriction of T to a given component must be a scalar, again by Schur's lemma. Hence $\mathcal{D}(X)^G$ is a subalgebra of the multiplication algebra on the set of isotypic components, and so is abelian. ■

There is a beautiful characterization of multiplicity-free representations in terms of invariant differential operators.

Theorem 2 ([10, Proposition 7.1]). *The algebra $\mathcal{D}(V)^G$ is commutative if and only if the representation (ρ, V) is multiplicity-free.*

Moreover, for multiplicity-free representations the algebra $\mathcal{D}(V)^G$ is isomorphic to a polynomial algebra, see [11] and [12].

The main purpose of this work is to obtain an analogous characterization for representations with spherical orbits.

4 Reduced actions and the algebra $\mathcal{Z}(X)^G$

Let X be an affine variety and G be a reductive algebraic group. The algebra $k[X]^G$ is finitely generated, and there is a canonical morphism $\pi_{X,G}$ (or just π_G) : $X \to X//G$, where $X//G$ is the affine variety corresponding to $k[X]^G$ and π_G^* is the inclusion $k[X]^G \subset k[X]$. The morphism π_G is surjective and induces a one-to-one correspondence between the closed G-orbits in X and the points of $X//G$, see [4, 4.4].

To any action $G : X$ one can canonically associate an action without non-constant invariants over some field of algebraic functions [13].

Namely, denote by K the field of quotients $Qk[X]^G$ of $k[X]^G$ and by $\overline{K}$ its algebraic closure. Let X^{red} be the spectrum of the $\overline{K}$-algebra $\overline{K}[X^{\mathrm{red}}] = \overline{K} \otimes_{k[X]^G} k[X]$. This is an irreducible affine variety over $\overline{K}$ defined over K, with $K[X^{\mathrm{red}}] = K \otimes_{k[X]^G} k[X]^G$. Its dimension equals

$$\dim X^{\mathrm{red}} = \dim X - \dim X//G, \qquad (1)$$

which is the dimension of a generic fiber of the quotient morphism $\pi_G : X \to X//G$.

The action of G on $k[X]$ is $k[X]^G$-linear and hence can be extended to an action of $G(K)$ on $K[X^{\mathrm{red}}]$, which, in its turn, can be extended to an action of $G(\overline{K})$ on $\overline{K}[X^{\mathrm{red}}]$. This gives rise to an action of $G(\overline{K})$ on X^{red} defined over K. This action is called *reduced action.*

Proposition 2. *The reduced action is spherical if and only if the following conditions hold:*

1) the action $G : X$ is an action with spherical orbits;

2) there exists an open dense subset $X_0 \subset X$ such that for any points $x_1, x_2 \in X_0$ with $Gx_1 \neq Gx_2$ there is $f \in k[X]^G$ such that $f(x_1) \neq f(x_2)$ (i.e. generic G-orbits can be separated by invariants).

Proof. We follow the proof of [13, Proposition 4]. Elements of $\overline{K}$ can be thought as algebraic functions on $Y = X//G$, and points of X^{red} as algebraic mappings $\phi : Y \to X$ such that $\pi_G \circ \phi = \mathrm{id}$. We may assume that $G \subset GL_n(k)$ and X is a G-invariant closed subvariety of k^n (see, e.g., [4]). Denote by $\mathfrak{b}$ a Borel subalgebra in the Lie algebra $\mathfrak{g}$ of the group G. Let us think elements of $\mathfrak{b}(\overline{K})$ as algebraic mappings $\xi : Y \to \mathfrak{b}$. The tangent algebra of the stabilizer $B(\overline{K})_\phi$ is defined by the linear equations

$$\xi(y)\phi(y) = 0 \qquad (2)$$

over $\overline{K}$. For a generic point $y \in Y$ they turn into linear equations defining the tangent algebra of the stabilizer $B_{\phi(y)}$ over k.

Obviously, the functional rank of system (2) is the maximum of the ranks of its specializations. Since all $\phi(Y)$ do not belong to a proper closed subvariety of X, we obtain that the dimension of a generic stabilizer for the action $B(\overline{K}) : X^{\mathrm{red}}$ is equal to that for the action $B : X$.

By Rosenlicht's theorem, the dimension of a generic B-orbit on X is equal to $\dim X -$ tr.deg $k(X)^B$. By (1), the action $G(\overline{K}) : X^{\mathrm{red}}$ is spherical if and only if

$$\text{tr.deg } k(X)^B = \dim X/\!/G = \text{tr.deg } Qk[X]^G. \tag{3}$$

Note that $Qk[X]^G \subseteq k(X)^G \subseteq k(X)^B$. Hence (3) is equivalent to tr.deg $Qk[X]^G =$ tr.deg $k(X)^G$ $=$ tr.deg $k(X)^B$. The second equality is the condition 1) of Proposition 2. By [4, 3.2], the first equality means that generic G-orbits can be separated by invariants. ■

Corollary 1. *Suppose that $G : V$ is a linear action of a semisimple group G. Then the reduced action $G(\overline{K}) : V^{\mathrm{red}}$ is spherical if and only if $G : V$ is an action with spherical orbits.*

Proof. By [4, Theorem 3.3], for a semisimple group action on a factorial variety the condition $Qk[X]^G = k(X)^G$ holds automatically. ■

Consider the centralizer of $k[X]^G$ in $\mathcal{D}(X)^G$:

$$\mathcal{Z}(X)^G = \left\{ D \in \mathcal{D}(X)^G \mid D(ab) = aD(b) \text{ for any } a \in k[X]^G,\ b \in k[X] \right\}.$$

Clearly, $k[X]^G \subset \mathcal{Z}(X)^G$. We are going to show that $\mathcal{Z}(X)^G$ contains differential operators of positive order.

There is a canonical morphism $(\pi_G)_* : \mathcal{D}(X)^G \to \mathcal{D}(X/\!/G)$, where $(\pi_G)_*(P)$ is the restriction of $P \in \mathcal{D}(X)^G$ to $k[X]^G = k[X/\!/G]$. We let $\mathcal{K}^n(X)$ denote the elements of $\mathcal{D}^n(X)$ which annihilate $k[X]^G$. Then, by definition, $\mathcal{K}^n(X)^G$ is the kernel of $(\pi_G)_*$ restricted to $\mathcal{D}^n(X)^G$, and $\mathcal{K}(X)^G := \cup_n \mathcal{K}^n(X)^G$ is the kernel of $(\pi_G)_*$. We have

$$0 \longrightarrow \mathcal{K}(X)^G \hookrightarrow \mathcal{D}(X)^G \xrightarrow{(\pi_G)_*} \mathcal{D}(X/\!/G).$$

Note that $\mathcal{D}^{n-1}(X)\tau(\mathfrak{g}) \subset \mathcal{K}^n(X)$, where $\tau(C)$ denotes the action of $C \in \mathfrak{g}$ on $k[X]$ as a derivation.

Define a positive integer n_0 by

$$\mathcal{K}^{n_0}(X)^G \neq 0 \text{ and } \mathcal{K}^m(X)^G = 0 \text{ for any } m < n_0.$$

Lemma 1. *The space $\mathcal{K}^{n_0}(X)^G$ is contained in $\mathcal{Z}(X)^G$.*

Proof. For any $a, b \in k[X]^G$ and $P \in \mathcal{K}^{n_0}(X)^G$ one has $[P, a](b) = P(ab) - bP(a) = 0$. Hence $[P, a] \in \mathcal{K}^{n_0-1}(X)^G$. By definition, this implies $[P, a] = 0$. ■

Now we are able to prove the main result of this note.

Theorem 3. *Let $G : X$ be an action with spherical orbits of a reductive group G on an affine variety X. Suppose that generic G-orbits can be separated by invariants. Then the algebra $\mathcal{Z}(X)^G$ is commutative.*

Proof. Elements of $\mathcal{Z}(X)^G$ commute with the $k[X]^G$-action on $k[X]$ and can be considered as differential operators on $K \otimes_{k[X]^G} k[X]$ or on $\overline{K} \otimes_{k[X]^G} k[X]$. Thus one has the embedding $\mathcal{Z}(X)^G \hookrightarrow \mathcal{D}(X^{\mathrm{red}})^{G(\overline{K})}$. By Propositions 1 and 2, the last algebra is commutative. ■

The algebra $\mathcal{D}(V)^G$ is the centralizer of its scalar subalgebra k. This algebra is commutative in spherical case. By Theorem 3, for representations of Table 1 (resp. Table 2) the commutativity holds if one replaces scalars by $k[q_1]$ (resp. $k[q_1, q_2]$).

Example 1. Let $G = (k^*)^s$ be an algebraic torus acting on $V = k^n$, $n \geq s$ by

$$(t_1, \ldots, t_s) * (x_1, \ldots, x_n) := (t_1x_1, \ldots, t_sx_s, x_{s+1}, \ldots, x_n).$$

It is clear that any torus action is an action with spherical orbits. For this particular action generic orbits can be separated by invariants. One has $k[V]^G = k[x_{s+1}, \ldots, x_n]$ and $\mathcal{Z}(V)^G = k[x_1\partial_1, \ldots, x_s\partial_s, x_{s+1}, \ldots, x_n]$.

Example 2. Consider the action $k^* : k^2$, $t * (x_1, x_2) = (tx_1, tx_2)$. This is an action with spherical orbits, but generic orbits can not be separated by invariants. Here $k[V]^G = k$ and $\mathcal{Z}(V)^G = \mathcal{D}(V)^G = k\langle x_1\partial_1, x_1\partial_2, x_2\partial_1, x_2\partial_2\rangle$. The last algebra is not commutative.

Finishing this section, we would like to state the following

Conjecture. *The following conditions are equivalent:*

1) an action $G : X$ is an action with spherical orbits and generic orbits can be separated by invariants;

2) the algebra $\mathcal{Z}(X)^G$ is commutative.

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