Hopf Bifurcations in Problems with $O(2)$ Symmetry: Canonical Coordinates Transformation

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Hopf bifurcations in problems with $O(2)$ symmetry are considered. In these problems, the Jacobian matrix is always singular at the circle of $\mathbb{Z}_2$ symmetric steady state solutions. While a couple of imaginary eigenvalue cross the imaginary axis, the Hopf bifurcation is not of standard type. The canonical coordinates transformation is used for removing the zero eigenvalue and converting the problem into the standard form. The method is applied to a system of ordinary differential equations on $\mathbb{C}^3$ with many parameters and the stable solutions are obtained using the centre manifold reduction. Further symmetry breaking bifurcation is obtained on periodic solutions, leading to modulated travelling waves solutions.

1 Introduction

We consider bifurcations which occur in systems with $O(2)$ symmetry. In particular we consider the Hopf bifurcation from a non-trivial steady state solutions giving rise to a branch of direction reversing wave (RW) solutions. Further bifurcation from these time periodic solutions lead to a branch of modulated travelling (MTW) solutions. The standard Hopf theorem [1] cannot be applied in this situation since there is a zero eigenvalue of the Jacobian at every nontrivial steady state solution, due to the group orbit of solutions.

Krupa [2] considers the related, but more general problem of bifurcation from group orbits for problems which are equivariant with respect to subgroups of $O(n)$. In this case, the degeneracy is dealt with by splitting the vector field into two parts, one tangent to the group orbit and one normal to it. A standard bifurcation analysis can then be performed on the normal vector field and the results are then interpreted for the whole vector field.

Barkley [3] considers Hopf bifurcation on the branch of travelling wave solutions, in the study of reaction diffusion system. He presented a low-dimensional ordinary differential equations model which has travelling wave solutions which undergo a Hopf bifurcation giving rise to MTW solutions. They decoupled some of the variables involved in the system by a simple change of coordinates to facilitate the analysis.

Landsberg and Knobeloch [4] studied the problem and showed that in problems with $O(2)$ symmetry a codimension-one symmetry breaking Hopf bifurcation from a circle of non-trivial steady states gives rise to periodic motions. These periodic solutions reverse their direction of propagation in a periodic manner. In another paper [5], they also refer to the modulated travelling waves which can bifurcate from the RW solutions. However, they did not perform any analysis of the bifurcations involved. We address these problems and related issues in this paper.

In this paper we first consider the method of canonical coordinates in more detail to give a clearer understanding of the type of solution which occurs and we analyse a further possible bifurcation from the branch of time periodic solutions, to modulated travelling wave solutions. In Section 2 we obtain the reduced equations to analyse the bifurcations and establish the relationship between different coordinate system employed. Section 3 is devoted to a numerical example to illustrate the method and to clarify the issues involved.

2 Setting the system

Consider the system of equations $\dot{z} = g(z, \lambda)$, where $z = (z_1, z_2, z_3) \in \mathbb{C}^3 =: X$ and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Let $z_j = x_j + iy_j$, $j = 1, 2, 3$ and assume that g is equivariant with respect to the diagonal action of $O(2)$ defined by

$$\begin{aligned} r_\alpha(z_1, z_2, z_3) &= \left(e^{i\alpha} z_1, e^{i\alpha} z_2, e^{i\alpha} z_3\right), \\ s(z_1, z_2, z_3) &= (\bar{z}_1, \bar{z}_2, \bar{z}_3)\,. \end{aligned} \tag{1}$$

Due to the reflection s the space X can be decomposed as $X = X^s \oplus X^a$, where X^s and X^a are the symmetric and anti-symmetric spaces with respect to the reflection s, respectively. Let us assume that non-trivial solutions $z_s = z_s(\lambda)$ bifurcate from trivial solutions at $\lambda = 0$. For these non-trivial steady states at least one of the variables, say z_1, is non-zero. There is a corresponding group orbit of solutions which are generated by the rotation. These solutions are contained in $\mathrm{Fix}(\mathbb{Z}_2) \times \mathbb{R}$, where $\mathbb{Z}_2 = \{I, s\}$. The reflection s implies that $y_1 = y_2 = y_3 = 0$. We now write the original equations in real form as

$$\dot{x} = f(x, \lambda), \qquad x = (x_1, x_2, x_3, y_1, y_2, y_3), \tag{2}$$

where $f = (f_1(x, \lambda), f_2(x, \lambda), f_3(x, \lambda), g_1(x, \lambda), g_2(x, \lambda), g_3(x, \lambda))$. The reflection s implies that $g_z(z_s, \lambda) = \mathrm{diag}\left(g_z^s(z_s, \lambda) : g_z^a(z_s, \lambda)\right)$, where g_z^s and g_z^a are associated with symmetric and anti symmetric spaces, respectively (see [6]). Clearly in real form these blocks take the form $g_z^s = [f_{ij}]$, $g_z^a = [g_{ij}]$, where $f_{ij} = \frac{\partial f_i}{\partial x_j}$ and $g_{ij} = \frac{\partial g_i}{\partial y_j}$, $i, j = 1, 2, 3$. All of the derivatives are evaluated at (z_s, λ).

The rotation symmetry implies that $g_z(z_s, \lambda) A z_s = 0$ for all λ, where the linear operator A is defined by $Az = \frac{d}{d\alpha}(r_\alpha z)|_{\alpha=0}$. In this case $Az = (iz_1, iz_2, iz_3)$ and so $Az_s \in X^a$. Thus the anti-symmetric block is singular. We now assume that $g_z(z_s, \lambda)$ also has eigenvalues $\pm i\omega_0$ at (z_0, λ_0), where $z_0 = z_s(\lambda_0)$. Since we are interested in symmetry breaking Hopf bifurcation then we assume that these eigenvalues occur in an anti-symmetric block. A necessary condition for this bifurcation is that the anti-symmetric block g_z^a has a minimum dimension three. We then show that a branch of periodic solutions bifurcates from the steady state branch at (z_0, λ_0) with a spatio-temporal symmetry (s, π). Further bifurcation can be obtained by breaking this symmetry. However, due to the zero eigenvalue the Hopf bifurcation is not of standard type. We thus use canonical coordinates [4, 7] in order to decouple one of the variables and then use the standard theory.

2.1 Reduced equations

We introduce the canonical coordinates transformation

$$w_1 = \frac{z_2}{z_1}, \qquad w_2 = \frac{z_3}{z_1}, \qquad r = |z_1|, \qquad \theta = \arg(z_1), \tag{3}$$

where $w_j = u_j + iv_j \in \mathbb{C}$, $j = 1, 2$ and $r \in \mathbb{R}$ are all invariant under the rotation and $\theta \to \theta + \alpha$. This enable us to decouple the θ variable from the others, when the system (2) is written in terms of these new variables, with the result

$$\dot{U} = G(U, \lambda), \tag{4}$$

$$\dot{\theta} = G_\theta(U, \lambda), \tag{5}$$

where $U = (u_1, u_2, r_1, v_1, v_2)$ and is invariant under the rotation. The reflection s acts as $s(u_1, u_2, r, v_1, v_2, \theta) = (u_1, u_2, r, -v_1, -v_2, -\theta)$. Thus, (4) has only a reflection symmetry. We note that, for the steady state problem $G(U,\lambda) = 0$, the symmetric space is characterized by $v_1 = v_2 = 0$ and the anti-symmetric space by $u_1 = u_2 = r = 0$. Thus for steady state solutions one can restrict the problem to the symmetric space and seek the solutions there. Now, let us $U_s = (u_1^s, u_2^s, r_1^s, v_1^s, v_2^s)$ be a steady solution of (4), then due to the reflection we can write $G_U = \operatorname{diag}(G_U^s(U_s,\lambda) : G_U^a(U_s,\lambda))$, where G_U^s is 3×3 matrix and G_U^a is 2×2. Expanding equation (4) explicitly, using the definition of canonical coordinates, system (2), all the steady state assumptions, and the computing algebra package MATHEMATICA we can be shown that G_U^a is given by

$$G_U^a = \begin{bmatrix} g_{22} - u_1^s g_{12} & g_{23} - u_1^s g_{13} \\ g_{32} - u_2^s g_{12} & g_{33} - u_2^s g_{13} \end{bmatrix}.$$

We will show that if $g_z^a(z_s,\lambda)$ has a pair of imaginary eigenvalues then G_U^a also does (see Theorem 1). Since G_U has eigenvalues $\pm i\omega_0$ and no zero eigenvalue therefore we can apply the standard theory, which implies that there exists a bifurcating branch of periodic solutions with $(s,\pi) \in \mathbb{Z}_2 \times S^1$ symmetry, since s and π both act as $-I$ on the eigenspace. It is possible to rescale time in order to have 2π-periodic solutions. This symmetry then implies that $-v_1(t+\pi) = v_1(t)$, $-v_2(t+\pi) = v_2(t)$, and $u_1(t+\pi) = u_1(t)$, $u_2(t+\pi) = u_2(t)$, $r(t+\pi) = r(t)$. As $s\theta = -\theta$, the equivariance condition related to equation (5) is

$$G_\theta(u_1, u_2, r, -v_1, -v_2, \lambda) = -G_\theta(u_1, u_2, r, v_1, v_2\lambda).$$

Thus, for the time periodic solutions, with $\tau = t + \pi$, we have

$$\begin{aligned} G_\theta(u_1(\tau), u_2(\tau), r(\tau), v_1(\tau), v_2(\tau), \lambda) &= G_\theta(u_1(t), u_2(t), r(t), -v_1(t), -v_2(t), \lambda) \\ &= -G_\theta(u_1(t), u_2(t), r(t), v_1(t), v_2(t), \lambda). \end{aligned}$$

Integrating the above equation over the interval $[0, 2\pi]$, considering 2π-periodicity of the functions u_1, u_2, r, v_1 and v_2, we obtain G_θ, hence $\dot\theta$ has zero mean, which implies that θ is periodic. Therefore, w_1 and w_2 are periodic and then the original variables z_1, z_2 and z_3 are also time periodic. A further bifurcation could occur from these periodic solutions which breaks the (s,π) symmetry. The theory related to this bifurcation is well developed [1] and this is a simple bifurcation on time periodic solutions which occurs in the reduced system (4). However, breaking this symmetry implies that $\dot\theta(t)$ has no longer zero mean and so we can write $\dot\theta(t) = c + \dot\theta_0(t)$ where $\dot\theta_0(t)$ has zero mean and c is constant. Hence $\theta(t) = ct + \theta_0(t) + k$, where k is the constant of integration that we set to zero. Since $\dot\theta_0$ has zero mean, hence θ_0 is periodic. Clearly, on the periodic solutions, due to (s,π) symmetry, $c = 0$ and therefore $\theta(t) = \theta_0(t)$ is periodic. However, if this symmetry is broken, then $c \neq 0$ and so θ is not periodic but is composed of a constant drift with velocity c superimposed on a periodic motion. This bifurcation arises as a simple symmetry breaking bifurcation in system (4). The solution in the original coordinates is then given by $z_1(t) = r(t)e^{i\theta(t)} = r(t)e^{i(ct+\theta_0(t))} = e^{i(ct)}\tilde z_1(t)$, where $\tilde z_1(t) = r(t)e^{i\theta_0(t)}$ is periodic. The first equation of (3) implies that $z_2(t) = e^{i(ct)}w_1(t)\tilde z_1(t) = e^{i(ct)}\tilde z_2(t)$, where $\tilde z_2(t) = w_1(t)\tilde z_1(t)$ is periodic. Finally, the second equation of (3) implies that $z_3(t) = e^{i(ct)}\tilde z_3(t)$, where $\tilde z_3(t)$ is periodic. Hence we have the solutions of the form

$$Z(t) = r_{ct} z(t), \tag{6}$$

where $z(t)$ is a periodic function of time. Thus $c = 0$ corresponds to a branch of periodic solutions while $c \neq 0$ corresponds to MTW solutions that consist of time periodic solutions drifting with constant velocity c along the group orbits. Note that the constant $k \neq 0$ simply gives rise to a one-parameter family of conjugate solutions, obtained by a constant rotation. Initially therefore, the solutions are oscillating with only a very small amount of drift and so the rotational motion, characterised by the variable θ, continues to oscillate. However, as the branch is followed further from the bifurcation point, the drift increases which could result in θ increasing (or decreasing) monotonically.

Now we show G_U^a also has a purely imaginary eigenvalues on the branch of non trivial steady state solutions. To see this, we first establish the relationship between two different coordinate systems.

2.2 Eigenvalues of the reduced system

We now consider the linearisation of equations (4) and (5) on the anti-symmetric space and obtain a connection between the two sets of coordinates in order to discuss about the eigenvalues of the reduced system. Since we have a standard Hopf bifurcation in (4) then the bifurcating solution near to the bifurcation point is given by $U(t) = U_s + \alpha\Phi(t) + O\left(\alpha^2\right)$, where $\Phi(t)$ is a solution of the linearisation of (4) about the steady state, i.e. $\Phi(t) = [0, 0, 0, V_1(t), V_2(t)]^T$, since it is the anti-symmetric component of G_U which has the imaginary eigenvalues. Again, near to the bifurcation point, we have $\theta = \theta^s + \alpha\Theta + O\left(\alpha^2\right)$, where Θ is the solution of the linearisation of (5) given by $\dot{\Theta} = \frac{\partial G_\theta}{\partial v_1}V_1 + \frac{\partial G_\theta}{\partial v_2}V_2$. It is easily shown that $\frac{\partial G_\theta}{\partial v_1} = g_{12}$ and $\frac{\partial G_\theta}{\partial v_2} = g_{13}$, evaluated at a symmetric steady state solution. The solution of this equation is $\Theta(t) = \Theta_0(t) + k$, where $\Theta_0(t)$ is periodic with zero mean and k is an arbitrary constant of integration. Thus, the linearisation of (4) and (5) about the symmetric steady state on the anti-symmetric space is given by

$$\dot{V}(t) = BV(t), \tag{7}$$

where B is a (3×3) matrix, constructed by the augmenting a third column and a third row to G_U^a. This consists of augmenting a column vector $[0, 0, 0]^T$ and a row vector $[g_{12}, g_{13}, 0]$. Note that eigenvalues of B are $\pm i\omega_0$ and zero, therefore the solution of (7) is $V_k(t) = [V_1(t), V_2(t), \Theta_0(t)]^T + k[0, 0, 1]^T = \tilde{V}(t) + ke_3$, where $k \in \mathbb{R}$ is arbitrary constant. Note that $\tilde{V}(t)$ is constructed from the complex eigenvectors of B corresponding to the eigenvalues $\pm i\omega_0$ and e_3 is the eigenvector corresponding to the zero eigenvalue. Converting back to the original coordinates, we have

$$z_1(t) = r(t)e^{i\theta(t)} = \left(r^s + O\left(\alpha^2\right)\right) e^{i\left(\theta^s + \alpha\Theta_0(t) + \alpha k + O\left(\alpha^2\right)\right)}.$$

Since $\sin\theta^s = 0$, this implies that

$$\begin{aligned}
x_1(t) &= r^s \cos\theta^s + O\left(\alpha^2\right) = x_1^s + O\left(\alpha^2\right), \\
y_1(t) &= \alpha\left(r^s \cos\theta^s\right)\left(\Theta_0(t) + k\right) + O\left(\alpha^2\right) = \alpha x_1^s(\Theta_0(t) + k) + O\left(\alpha^2\right).
\end{aligned}$$

Similarly, it can be shown, by using definition of canonical coordinates, that

$$\begin{aligned}
x_2(t) &= x_2^s + O\left(\alpha^2\right), \\
x_3(t) &= x_3^s + O\left(\alpha^2\right), \\
y_2(t) &= \alpha\left[x_2^s(\Theta_0(t) + k) + x_1^s V_1(t)\right] + O\left(\alpha^2\right), \\
y_3(t) &= \alpha\left[x_3^s(\Theta_0(t) + k) + x_1^s V_2(t)\right] + O\left(\alpha^2\right).
\end{aligned}$$

Hence, on the anti-symmetric space, the linearisation of the original equations given by

$$\dot{Y}(t) = g_z^a(z_0, \lambda_0)Y(t), \tag{8}$$

has solutions of the form

$$Y_k(t) = \begin{bmatrix} x_1^s(\Theta_0(t) + k) \\ x_2^s(\Theta_0(t) + k) + x_1^s V_1(t) \\ x_3^s(\Theta_0(t) + k) + x_1^s V_2(t) \end{bmatrix} = T(\tilde{V}(t) + ke_3) = TV_k(t),$$

where $T = \begin{bmatrix} 0 & 0 & x_1^s \\ x_1^s & 0 & x_2^s \\ 0 & x_1^s & x_3^s \end{bmatrix}$. Note that $Te_3 = [x_1^s, x_2^s, x_3^s]^T = Az_0^a$, where Az_0^a is the anti-symmetric part of Az_0, which is a solution of (8) since it is independent of time and $g_z^a(z_0, \lambda_0)Az_0^a = 0$. A more precise result is the following.

Theorem 1. *If g_z^a is the anti-symmetric block with the eigenvalues $\pm i\omega_0$ and 0 then*

(i) $g_z^a(z_0, \lambda_0) = TBT^{-1}$,

(ii) $V_k(t)$ is a solution of (7) if and only if $Y_k(t) = TV_k(t)$ is a solution of (8).

Proof. (i) Note that the last column of T is $[x_1^s, x_2^s, x_3^s]^T = Az_0^a$. Thus

$$g_z^a(z_0, \lambda_0)T = x_1^s \begin{bmatrix} g_{12} & g_{13} & 0 \\ g_{22} & g_{23} & 0 \\ g_{32} & g_{33} & 0 \end{bmatrix},$$

and then it is easily verified that $T^{-1}g_z^a(z_0, \lambda_0)T = B$.

(ii) follows immediately from (i). ■

The first of these results show more clearly that when g_z^a has eigenvalues $\pm i\omega_0$, then so does G_U^a. The second presents the relationship between the eigenvectors.

3 A numerical example

3.1 An equation on $\mathbb{C}^3$

Consider the system [4],

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ \dot{z}_3 &= \lambda z_1 + \nu z_2 + \eta z_3 + a|z_1|^2 z_1 + b|z_2|^2 z_1 + c|z_1|^2 z_2 + d|z_1|^2 z_3 \\ &\quad + e z_1^2 \bar{z}_2 + f z_1^2 \bar{z}_3 + g z_2^2 \bar{z}_1 + h|z_2|^2 z_2 + j z_1 \bar{z}_2 z_3 + k z_1 z_2 \bar{z}_3 + l|z_2|^2 z_3 + m z_2^2 \bar{z}_3. \end{aligned} \tag{9}$$

These equations are the normal form for a triple zero bifurcation with group $O(2)$ symmetry and such have a number of applications, particularly in fluid dynamics [9]. We write these equations as $\dot{z} = g(z, \lambda)$, where $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ and λ is regarded as the bifurcation parameter. It is easily verified that this system is equivariant with respect to the diagonal action of $O(2)$ defined by (1) (see [11]). System (9) has trivial solution $z = 0$ for $\forall\ \lambda$; a bifurcating branch of solutions occurs at $\lambda = 0$, and is given by $x_1^2 = -\lambda/a$, $y_1 = y_2 = y_3 = 0$. As these solutions are invariant under the reflection symmetry s, conjugate solutions are obtained by applying the rotational operator r_α, giving rise to a circle of steady state solutions for each λ. The trivial solutions will be stable for $\lambda < 0$ and unstable for $\lambda > 0$, if $\eta, \nu < 0$. The bifurcating branch will then be stable if it is supercritical. This occurs if $a < 0$. Therefore we choose $a = -4.0$, $\eta = -2.5$, $\mu = -10$ so that a supercritical bifurcation occurs at $\lambda = 0$. In order to have a couple of imaginary eignvalue in anti-symmetric block, evaluated at a non-trivial steady state, we choose $d = 0$, $f = -16$, $c = -1$, $e = 0$. Therefore a symmetry breaking Hopf bifurcation occur at $\lambda = 2.5$, giving rise to a branch of RW's. We choose the rest of the parameters so that periodic orbits are stable. This can be carried out using the canonical coordinates transformation followed by the centre manifold reduction [10]. We omit the discussion and only introduce the rest of the parameters as: $b = -10$, $g = 30$, $h = 1$, $j = 10$, $k = -30$, $l = 20$, $m = -30$. For these

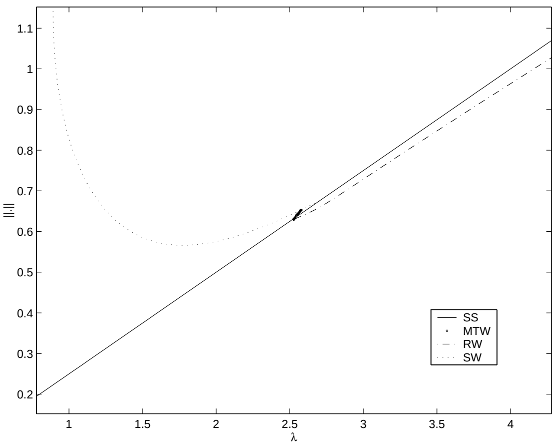

Figure 1. Bifurcation diagram of the equations (9). With $\eta = -2.5$ a branch of RW solutions appear at $\lambda = 2.5$, and the SW solutions occur at $\lambda = 2.7$. A secondary bifurcation occurs on the branch of SW solutions at $\lambda = 2.577$ giving rise to a branch of MTW solutions which connects to RW solutions at $\lambda = 2.526$.

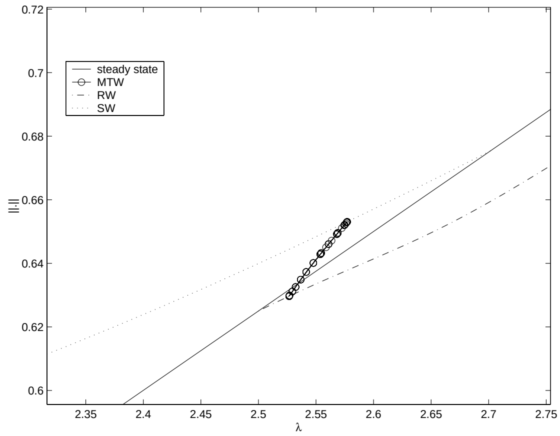

Figure 2. Enlarged bifurcation diagram around the bifurcation points. All bifurcation points stated in Fig. 1 can be seen clearly. A branch of MTW solutions connects two branches of periodic orbits. A torus bifurcation obtained on this branch at $\eta = -2.5$ and $\lambda = 2.547$.

values of the parameters we applied the numerical method described in the previous section and the following bifurcations are obtained: on the branch of non-trivial steady states a symmetry preserving Hopf bifurcation leading to a branch of periodic orbits occurs at $\lambda = 2.70$, these are standing waves (SW) which lie in symmetric space. On the branch of RW solutions a secondary bifurcation, giving rise to a branch of MTW solutions, occurs at $\lambda = 2.526$. This branch connects with the branch of SW's at $\lambda = 2.577$. On the branch of the MTW solutions a torus bifurcation is also obtained at $\lambda = 2.547$. A bifurcation diagram of these solutions is shown in Fig. 1. This diagram is enlarged in Fig. 2 to give a clear picture of the bifurcation points involved in the problem. For $\lambda = 2.595$ and $\eta = -2.50$ a RW is given in Fig. 3, this is a time periodic solution and reverses its direction of propagation in a periodic manner [4]. In Fig. 4 and Fig. 5, the MTW solutions are represented for different values of λ and η. All of these solutions were obtained using the package AUTO [8]. The MTW's were reconstructed using equation (6).

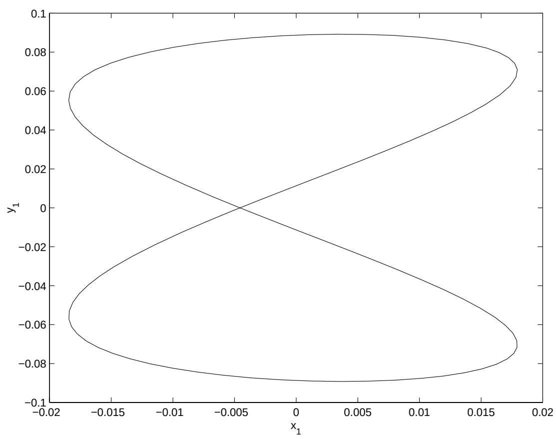

Figure 3. A RW solution at $\lambda = 2.595$ and $\eta = -2.50$. This is a time periodic solution with the spatio-temporal symmetry.

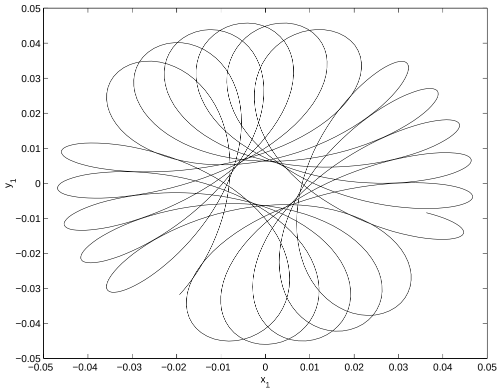

Figure 4. A MTW solution at $\lambda = 2.531$ and $\eta = -2.50$. There is a 4-petal flower (2 small and two large) repeating itself with time progression.

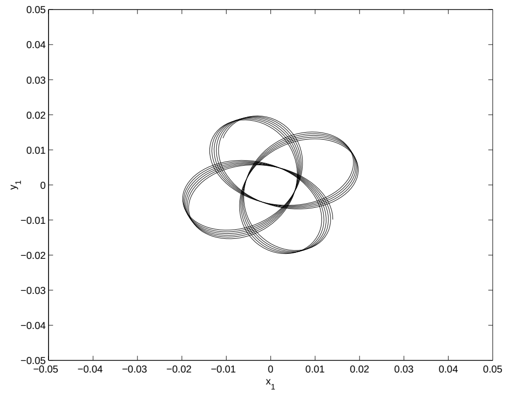

Figure 5. A MTW solution at $\lambda = 2.0886$ and $\eta = -2.0800$. There is a 4-petal flower (each petal has same amplitude) repeating itself with time progression.

4 Conclusions

The Hopf bifurcation in problems with $O(2)$ symmetry is considered. The canonical coordinates transformation were used in order to analyse the problem using standard theory, and also to convert the solutions back into the original coordinates in order to obtain a correct interpretation of the results.

We obtained time periodic solutions with spatio-temporal symmetry. Further bifurcation is obtained by breaking this symmetry resulting in MTW solutions, which is due to the fact that one of the variables in canonical coordinates drift with constant velocity. In addition an example on $\mathbb{C}^3$ with many parameters [4] was considered to clarify the analysis and centre manifold reduction is used to obtain stable solutions. Two Hopf bifurcations leading to the SW's and the RW's were obtained on steady state solutions. The occurrence of a second Hopf bifurcation indicates that if a second parameter was varied there may be a Hopf/Hopf mode interaction. This is the case that considered by Amdjadi [11] who introduced a numerical method for such mode interactions.

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