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INVESTIGATION OF FORCED OSCILLATIONS IN CIRCULAR CYLINDRICAL VESSELS SEPARATED BY DIAMETRICAL BARRIERS

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We consider nonlinear oscillations of an ideal incompressible liquid in a partially filled vertical semicircular cylindrical tank. We construct approximate periodic solutions for a four-mode system that describes nonlinear oscillations in a semicircular cylindrical tank under the action of a perturbation force in the plane of the barrier. We construct and investigate the domains of stability and instability for the physical processes considered. We perform a numerical realization of the method and analyze the hydrodynamic interaction of the liquid with the tank. The problem considered is of interest for the investigation of nonlinear processes in a liquid in the case of tanks with diametrical barriers.

Introduction

To decrease the negative influence of oscillations of the free surface of a liquid on the stability of motion of a "body–liquid" system, various structural devices are used in practice. In particular, devices in the form of rigid or elastic barriers are widely used for this purpose. Barriers in vessels considerably affect the interaction between the body and the liquid. This problem is well studied in the case of linear statements of the problem of dynamics of solid bodies with liquid [1, 5, 8, 9, 20, 21]. However, this leads to a series of paradoxes, which can only be eliminated by considering the corresponding problems in a nonlinear statement. This is confirmed by numerous experimental [1, 8, 9, 19] and theoretical [10, 11, 15, 16, 17, 18] results.

Problems of nonlinear oscillations of a liquid are mainly based on the potential theory of an ideal incompressible liquid. At present, analytic and numerical–analytic methods based mainly on asymptotic and modal techniques are developed. In recent years, a modal approach based on the variational principles of mechanics [2, 14, 22] has been widely used. According to this approach, an original problem for partial differential equations is reduced to systems of nonlinear ordinary differential equations containing time-dependent parameters that characterize the evolution of the free surface of the liquid. This approach has numerous significant advantages over analytic methods based on the principles of the theory of perturbations [3, 5, 12, 23]. The contemporary state of mathematical problems of the nonlinear theory of oscillations of a liquid in moving vessels is discussed in [24].

In the present paper, we present the results of theoretical investigations of the behavior of a liquid in a moving vessel in the form of a circular cylindrical tank separated by a diametrical barrier into two parts. These investigations are based on the nonlinear mathematical model of motion of a liquid constructed in [4] by using the Miles–Lukovs'kyi method.

1. Mathematical Statement of the Problem

We consider the translational motion of a solid body that has the form of a vertical semicircular cylinder containing a bounded amount of an ideal incompressible liquid with density ρ. In what follows, we assume that the walls of the cylinder are absolutely rigid. We describe this motion in the cylindrical coordinate system x, ξ, η associated with the cylinder and choose its origin to lie on the unperturbed free surface Σ_0. The Ox-axis is directed along the axis of the cylinder in the direction opposite to the vector of acceleration of the Earth's gravity $\vec{g}$.

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We restrict ourselves to the consideration of irrotational motions of the liquid. Assume that the distribution of the velocities of the liquid can be represented as the gradient of a potential function $\Phi(x, \xi, \eta, t)$, namely,

$$\vec{v} = \nabla\Phi(x, \xi, \eta, t). \tag{1}$$

Furthermore, the velocity potential must be a solution of the following nonlinear boundary-value problem with free boundary, which (according to [3]) associates $\Phi(x, \xi, \eta, t)$ with the instantaneous location of the free surface, the form of which is given by the equation $\zeta(x, \xi, \eta, t) = 0$:

$$\triangle\Phi = 0, \quad \vec{r} \in Q, \tag{2}$$

$$\frac{\partial\Phi}{\partial\nu} = \vec{v}_0 \cdot \vec{\nu}, \quad \vec{r} \in S, \tag{3}$$

$$\frac{\partial\Phi}{\partial\nu} = \vec{v}_0 \cdot \vec{\nu} - \frac{\zeta_t}{|\nabla\zeta|^2}, \quad \vec{r} \in \Sigma, \tag{4}$$

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi, \nabla\Phi) - \nabla\Phi \cdot \vec{v}_0 + U = 0, \quad \vec{r} \in \Sigma, \tag{5}$$

where $\vec{\nu}$ is the unit vector of the outer normal to the surface of the domain Q occupied by the liquid, S is the solid wall (including the surface of the barrier), Σ is the perturbed free surface of the liquid, $\vec{r}$ is the radius vector of a point of the volume of the liquid Q in the associated coordinate system, $\vec{v}_0$ is the vector of translational motion of the volume of the liquid Q, and U is the potential of the gravity forces.

The distribution of pressure in the volume of the liquid is determined by the Lagrange–Cauchy integral in the cylindrical coordinate system $Ox\xi\eta$:

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi, \nabla\Phi) - \nabla\Phi \cdot \vec{v}_0 + gx + \frac{p - p_0}{\rho} = 0, \tag{6}$$

where p_0 is the pressure of gas above the free surface of the liquid under the condition of conservation of the volume of the liquid

$$\int\limits_{Q(t)} dQ = 0. \tag{7}$$

Assume that pressure p on Σ is constant, i.e., $p_0 = \text{const}$. The condition of conservation of volume (7) is the condition of solvability of the Neumann boundary-value problem (2)–(4). The evolution problem with free boundary (2)–(5) should be complemented with Cauchy initial conditions related to the initial profile of the free surface $\Sigma(t_0)$ and the distribution of velocities at the initial moment of time $t = t_0$ on it, namely,

$$\zeta(x, \xi, \eta, t_0) = \zeta_0(x, \xi, \eta), \qquad \left.\frac{\partial\Phi}{\partial\nu}\right|_{\Sigma(t_0)} = \Phi_0(x, \xi, \eta), \tag{8}$$

where $\zeta_0(x, \xi, \eta)$ and $\Phi_0(x, \xi, \eta)$ are known functions.

2. Modal System

For the generalized solutions of the boundary-value problem (2)–(5), the functional

$$W = \int_{t_1}^{t_2} L dt \tag{9}$$

is stationary [3, 13, 14], i.e.,

$$\delta W = \delta \int_{t_1}^{t_2} L dt = 0, \tag{10}$$

where

$$L = \int_{Q(t)} p dQ = -\rho \int_{t_1}^{t_2} \int_{Q(t)} \left[\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla\Phi, \nabla\Phi) - \nabla\Phi \cdot \vec{v}_0 + gx \right] dQ. \tag{11}$$

To determine the free surface and the velocity potential in the given volume, we solve the variational problem (10) by the Miles–Lukovs'kyi method [2, 14]. According to this method, the form of the free surface (under the assumption of solvability with respect to one variable) $x = f(\xi, \eta, t)$ and the velocity potential $\Phi(x, \xi, \eta, t)$ are represented in the form of a Fourier series in a certain complete system of orthogonal functions:

$$f(\xi, \eta, t) = \sum_{i=1}^{\infty} \beta_i(t) f_i(\xi, \eta), \tag{12}$$

$$\Phi(x, \xi, \eta, t) = \vec{v}_0 \cdot \vec{r} + \sum_{j=1}^{\infty} R_j(t) \varphi_j(x, \xi, \eta), \tag{13}$$

where $f_i(\xi, \eta)$ is a complete system of functions in the Hilbert space $L_2(\Sigma_0)$ orthogonal to a constant and given on the unperturbed free surface Σ_0, $\beta_i(t)$ are generalized Fourier coefficients that depend on time (as a parameter) and are generalized coordinates that characterize the deviation of the free surface of the liquid from the unperturbed location, $R_j(t)$ are parameters that characterize the variation in the velocity potential with time, and $\varphi_j(x, \xi, \eta)$ is the system of harmonic functions in the domain $Q(t)$ that satisfy the boundary nonflow condition on the wetted surface $S(t)$.

We substitute expansion (13) for the velocity potential in relation (11), taking expansion (12) into account. Integrating the "Lagrange function" L in the above-mentioned variational principle with respect to the space variables, we represent it as a function of the variables $\beta_i(t)$, $R_j(t)$, and $\dot{R}_j(t)$.

For the determination of the parameters $\beta_i(t)$ and $R_j(t)$, we deduce from (10) the following general system of nonlinear ordinary differential equations:

$$\frac{\partial L}{\partial \beta_i} = 0, \quad i = 1, 2, \ldots,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{R}_n} - \frac{\partial L}{\partial R_n} = 0, \quad n = 1, 2, \ldots. \tag{14}$$

Results convenient for practical application can be obtained by taking into account finitely many parameters $\beta_i(t)$ and $R_j(t)$ and selecting among them the parameters that play the leading role. For the analytic realization of the method presented, one should also impose certain restrictions on the order of smallness of these parameters. In what follows, among the entire set of $\beta_i(t)$, we take into account only four coefficients in (12), namely, $\beta_0(t)$, $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$, which make a substantial contribution to the associated masses and moments of inertia of the "body–liquid" system within the framework of the linear theory. For convenience, we redenote them as follows:

$$\beta_0 = p_0, \qquad \beta_1 = r_1, \qquad \beta_2 = p_2, \qquad \beta_0 = r_3.$$

In the present paper, assuming that $r_1 \approx \epsilon$, $p_0 \approx p_2 \approx \epsilon^2$, and $r_3 \approx \epsilon^3$, we consider a mathematical model with parameters of order up to ϵ^3 inclusive.

3. Forced Oscillations in a Compartment under the Action of a Harmonic Perturbation Force

In the present paper, we consider only oscillations of a compartment under the action of a perturbation force in the plane of the diametrical barrier. In [4], the method presented in [3] was applied to a circular cylinder, and the following system of nonlinear ordinary differential equations describing oscillations of a liquid in the given volume was deduced from (14):

$$L_0(r_1, p_0) = \ddot{p}_0 + \sigma_0^2 p_0 + d_{14}^* r_1 \ddot{r}_1 + d_8^* \dot{r}_1^2 = 0,$$

$$L_1(r_1, p_0, p_2) = \ddot{r}_1 + \sigma_1^2 r_1 + d_1^*(r_1 \dot{r}_1^2 + r_1^2 \ddot{r}_1) + d_3^*(p_2 \ddot{r}_1 + \dot{p}_2 \dot{r}_1) + d_4^* r_1 \ddot{p}_2$$
$$+ d_5^*(\dot{p}_0 \dot{r}_1 + p_0 \ddot{r}_1) + d_6^* r_1 \ddot{p}_0 + P_1 \omega^2 \cos(\omega t) = 0,$$

$$L_2(r_1, p_2) = \ddot{p}_2 + \sigma_2^2 p_2 + d_{15}^* r_1 \ddot{r}_1 + d_7^* \dot{r}_1^2 = 0,$$

$$L_3(r_1, p_2, r_3) = \ddot{r}_3 + \sigma_3^2 r_3 + d_9^* r_1 \dot{r}_1^2 + d_{10}^* r_1^2 \ddot{r}_1 + d_{11}^* \dot{r}_1 \dot{p}_2 + d_{12}^* p_2 \ddot{r}_1 + d_{13}^* r_1 \ddot{p}_2 = 0, \tag{15}$$

where

$$
\begin{gathered}
d_1^* = \frac{d_1}{\mu_1}, \qquad d_3^* = \frac{d_3}{\mu_1}, \qquad d_4^* = \frac{d_4}{\mu_1}, \qquad d_5^* = \frac{d_5}{\mu_1}, \qquad d_6^* = \frac{d_6}{\mu_1}, \\
d_7^* = \frac{d_7}{\mu_2}, \qquad d_8^* = \frac{d_8}{\mu_0}, \qquad d_9^* = \frac{d_9}{\mu_3}, \qquad d_{10}^* = \frac{d_{10}}{\mu_3}, \qquad d_{11}^* = \frac{d_{11}}{\mu_3}, \\
d_{12}^* = \frac{d_{12}}{\mu_3}, \qquad d_{13}^* = \frac{d_{13}}{\mu_3}, \qquad d_{14}^* = \frac{d_6}{\mu_0}, \qquad d_{15}^* = \frac{d_4}{\mu_2}, \qquad P_1 = \frac{H\lambda_{23}}{\mu_1}.
\end{gathered} \tag{16}
$$

In what follows, we omit the asterisk $*$ in the coefficients d_i.

It is necessary to find periodic solutions of system (15). To determine periodic solutions of this system, we represent the generalized coordinate $r_1(t)$ in the form of a finite Fourier series with unknown coefficients [6]:

$$
r_1(t) = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos k\omega t + \beta_k \sin k\omega t). \tag{17}
$$

In what follows, we preserve only the first harmonics in this approximation:

$$
r_1(t) = A\cos\omega t + B\sin\omega t. \tag{18}
$$

Using the first equation and the last two equations of system (15), which are linear with respect to $p_0(t)$, $p_2(t)$, and $r_3(t)$, we obtain explicit relations for the generalized coordinates $p_0(t)$, $p_2(t)$, and $r_3(t)$:

$$
\begin{gathered}
p_0(t) = (A^2 + B^2) f_0 + (A^2 - B^2) f_2 \cos 2\omega t + 2AB f_2 \sin 2\omega t, \\
p_2(t) = (A^2 + B^2) g_0 + (A^2 - B^2) g_2 \cos 2\omega t + 2AB g_2 \sin 2\omega t, \\
r_3(t) = (A^3 + AB^2) h_1 \cos\omega t + (B^3 + A^2 B) h_1 \sin\omega t \\
+ (A^3 - 3AB^2) h_3 \cos 3\omega t + (3A^2 B - B^3) h_3 \sin 3\omega t,
\end{gathered} \tag{19}
$$

where

$$
\begin{gathered}
f_0 = \frac{d_{14} - d_8}{2\bar\sigma_0^2}, \quad f_2 = \frac{d_{14} + d_8}{2(\bar\sigma_0^2 - 4)}, \quad g_0 = \frac{d_{15} - d_7}{2\bar\sigma_2^2}, \quad g_2 = \frac{d_{15} + d_7}{2(\bar\sigma_2^2 - 4)}, \\
h_1 = \frac{1}{4(\bar\sigma_3^2 - 1)} (-d_9 + 3d_{10} + 4d_{12} g_0 + 2(-2d_{11} + d_{12} + 4d_{13}) g_2), \\
h_3 = \frac{1}{4(\bar\sigma_3^2 - 9)} (d_9 + d_{10} + 2(2d_{11} + d_{12} + 4d_{13}) g_2), \\
\bar\sigma_0^2 = \frac{\sigma_0^2}{\omega^2}, \quad \bar\sigma_1^2 = \frac{\sigma_1^2}{\omega^2}, \quad \bar\sigma_2^2 = \frac{\sigma_2^2}{\omega^2}, \quad \bar\sigma_3^2 = \frac{\sigma_3^2}{\omega^2}.
\end{gathered} \tag{20}
$$

Substituting relations (18) and (19) in the Bubnov–Galerkin equations

$$\int_0^{\frac{2\pi}{\omega}} L_1(p_0, r_1, p_2)\cos\omega t dt = 0,$$
$$\int_0^{\frac{2\pi}{\omega}} L_1(p_0, r_1, p_2)\sin\omega t dt = 0, \tag{21}$$

we obtain a system of algebraic equations for the determination of the amplitudes A and B:

$$A(\bar{\sigma}_1^2 - 1) + A^3 m_1 + AB^2 m_1 + P_1 = 0,$$
$$B(\bar{\sigma}_1^2 - 1) + B^3 m_1 + A^2 B m_1 = 0, \tag{22}$$

where

$$m_1 = -\frac{d_1}{2} - d_5 f_0 - d_3 g_0 + \left(2d_6 - \frac{d_5}{2}\right) f_2 + \left(2d_4 - \frac{d_3}{2}\right) g_2. \tag{23}$$

If higher harmonics ($n > 1$) are taken into account in series (17), then expressions (19) take a more complex form. Correspondingly, the number of conditions (21) increases (to $2n+1$) and the number of Eqs. (22) increases as well. However, in the case of a circular cylindrical cavity without barriers, this improves the final result neither qualitatively nor quantitatively [3].

Analyzing system (22) for $P_1 \neq 0$, we get

$$A \neq 0, \qquad B = 0.$$

Taking relations (18), (19), and (22) into account, we conclude that the nonlinear system (15) may have only the following approximate periodic solution:

$$r_1(t) = A\cos\omega t,$$
$$p_0(t) = A^2 f_0 + A^2 f_2 \cos 2\omega t,$$
$$p_2(t) = A^2 g_0 + A^2 g_2 \cos 2\omega t, \tag{24}$$
$$r_3(t) = A^3 h_1 \cos\omega t + A^3 h_3 \cos 3\omega t,$$

where the amplitude A is determined from the cubic equation

$$A^3 m_1 + A(\bar{\sigma}_1^2 - 1) + P_1 = 0. \tag{25}$$

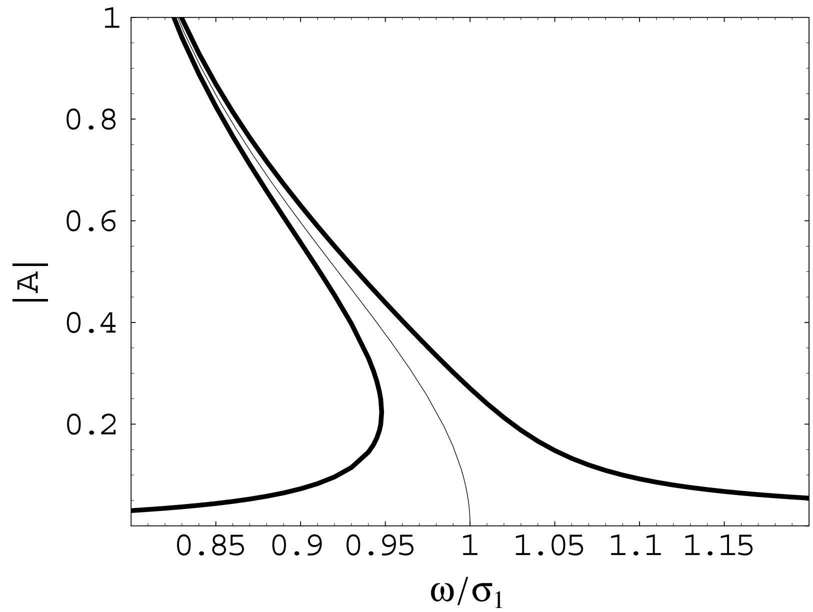

Fig. 1. Amplitude–frequency characteristics of oscillations of the liquid in a semicircular cylindrical tank $h/d = 1$ in the neighborhood of the principal resonance with the parameter $\chi_{11} = k_{11} th k_{11} h,$ ($k_{11} = 1.8412$).

By using other methods of nonlinear mechanics, e.g., the Krylov–Bogolyubov–Mitropol'skii method or the method of harmonic balance, one can obtain analogous equations for the amplitude-frequency characteristics of a liquid.

The dependence of the modulus of amplitude of forced oscillations of a liquid (25) on the parameters P_1 and ω is displayed in Fig. 1. Setting $P_1 = 0$ in Eq. (25), we obtain an equation for the determination of the dependence of the amplitudes of free oscillations of the liquid on the frequency (a so-called skeleton line). In Fig. 1, it is depicted by a thin line.

The analysis of the amplitude–frequency characteristics for the free oscillations of a liquid in a semicylindrical tank shows that, for the dynamical system considered, a nonlinearity of the "soft" or "rigid" type takes place, depending on the depth of the liquid h. The critical value of dimensionless depth h_*/R for which the reverse of frequencies occurs is equal to 0.597, whereas for a cylinder that is not partitioned into compartments we have $h_*/R = 0.52239$.

4. Investigation of Stability of Periodic Solutions of the System of Equations (15)

Periodic oscillations described by relations (24) are not always realized in practice. Actually, only stable motions take place.

To investigate the problem of dynamic stability of the free surface of a liquid, we analyze solutions of variational equations.

We deduce variational equations for oscillations in the case where Lyapunov-unperturbed motions of system (15) are described by relations (24). Parallel with unperturbed motions (marked below by ˜), we consider motions close to them of the form

$$r_1(t) = \tilde{r}_1(t) + \alpha(t), \qquad p_0(t) = \tilde{p}_0(t) + \beta(t),$$
$$p_2(t) = \tilde{p}_2(t) + \gamma(t), \qquad r_3(t) = \tilde{r}_3(t) + \delta(t). \tag{26}$$

To obtain variational equations for the perturbations $\alpha,\ \beta,\ \gamma,$ and $\delta,$ we substitute the perturbed solutions (26) in system (15) and take into account that the unperturbed solution (24) satisfies this system. Linearizing the obtained system with respect to perturbations, we obtain the following system of variational equations:

$$L_0 = \ddot{\beta}(t) + \sigma_0^2\beta(t) + 2d_8\dot{\alpha}(t)\dot{r}_1(t) + d_{14}(r_1(t)\ddot{\alpha}(t) + \alpha(t)\ddot{r}_1(t)),$$
$$L_1 = (1 + d_5p_0(t) + d_3p_2(t) + d_1r_1^2(t))\ddot{\alpha}(t) + \dot{\alpha}(t)(d_5\dot{p}_0(t) + d_3\dot{p}_2(t) + 2d_1r_1(t)\dot{r}_1(t))$$
$$+ \alpha(t)(\sigma_1^2 + d_1\dot{r}_1(t)^2 + d_6\ddot{p}_0(t) + d_4\ddot{p}_2(t) + 2d_1r_1(t)\ddot{r}_1) + d_6r_1(t)\ddot{\beta}(t) + d_4r_1(t)\ddot{\gamma}(t)$$
$$+ d_5\dot{\beta}(t)\dot{r}_1(t) + d_3\dot{\gamma}(t)\dot{r}_1(t) + d_5\beta(t)\ddot{r}_1(t) + d_3\gamma(t)\ddot{r}_1(t), \tag{27}$$
$$L_2 = \ddot{\gamma}(t) + \sigma_2^2\gamma(t) + 2d_7\dot{\alpha}(t)\dot{r}_1(t) + d_{15}(r_1(t)\ddot{\alpha}(t) + \alpha(t)\ddot{r}_1(t)),$$
$$L_3 = \ddot{\delta}(t) + \sigma_3^2\delta(t) + (d_{12}p_2(t) + d_{10}r_1^2(t))\ddot{\alpha}(t) + \dot{\alpha}(t)(d_{11}\dot{p}_2(t) + 2d_9r_1(t)\dot{r}_1(t))$$
$$+ \alpha(t)(d_9\dot{r}_1(t)^2 + d_{13}\ddot{p}_2(t) + 2d_{10}r_1(t)\ddot{r}_1) + d_{11}\dot{\gamma}(t)\dot{r}_1(t) + d_{12}\gamma(t)\ddot{r}_1(t) + d_{13}\ddot{\gamma}(t)r_1(t).$$

The variational equations (27) are linear equations with periodic coefficients. The Floquet theory describes the main properties of solutions of these equations. In the literature, these equations are called equations of the Hill type. Solutions of these equations are classified into three groups, namely, (i) “unstable” solutions, which infinitely increase as $t \to \infty,$ (ii) “stable” solutions, which remain bounded as $t \to \infty,$ and (iii) solutions with period T or $2T,$ which are called neutral (they are considered as a special case of stable solutions).

Unstable solutions occupy certain domains on the plane of parameters of these equations. Furthermore, the domains of instability are separated from the domains of stability by periodic solutions with periods T and $2T.$ Two solutions of the same period bound a domain of instability, and two solutions of different periods bound a domain of stability. Thus, the problem of determination of the boundaries of the domains of instability reduces to the determination of conditions under which the differential equation has solutions with periods T and $2T.$

Therefore, the problem of investigation of the stability of the periodic solutions (24) reduces to the investigation of solutions of system (27). The equations obtained are a system of equations with periodic coefficients and, by virtue of the Floquet–Lyapunov theorem, the fundamental system of their solutions contains solutions of the form

$$\alpha(t) = e^{\lambda t}\varphi_1(t), \qquad \beta(t) = e^{\lambda t}\varphi_2(t),$$
$$\gamma(t) = e^{\lambda t}\varphi_3(t), \qquad \delta(t) = e^{\lambda t}\varphi_4(t), \tag{28}$$

where λ is the characteristic exponent of the system and φ_i are $2\pi/\omega$ -periodic functions.

It follows from relations (28) that the stability of solutions (24) depends on the values of the characteristic exponent λ. If all characteristic exponents have negative real parts, then the periodic solutions are stable. If, among the characteristic exponents, there is at least one with positive real part, then the periodic solutions are unstable. The case where the real part of the characteristic exponent is equal to zero is more complicated. If the characteristic exponents are simple or multiple with simple elementary divisor, then the solutions of the system of variational equations (27) are bounded in time.

To derive an equation for the determination of characteristic exponents, we represent the periodic function $\varphi_1(t)$ in the form of a Fourier series and retain only the first harmonics in the expansion, i.e.,

$$\varphi_1(t) = a_1 \cos\omega t + b_1 \sin\omega t, \tag{29}$$

where a_1 and b_1 are certain constant coefficients.

We substitute relations (29) and (28) into the system of variational equations (27). One can explicitly determine the perturbations $\beta(t)$, $\gamma(t)$, and $\delta(t)$ from the first equation and the last two equations of the system and express them in terms of a_1 and b_1.

For the determination of the coefficients a_1 and b_1, we obtain the following homogeneous system of linear algebraic equations:

$$\begin{aligned} C_{11}a_1 + C_{12}b_1 &= 0, \\ C_{21}a_1 + C_{22}b_1 &= 0. \end{aligned} \tag{30}$$

Denote the ratio λ/ω by $\bar\lambda$. The coefficients of the linear algebraic system (30) C_{11}, C_{12}, C_{21}, and C_{22} are expressed in terms of the coefficients of the system of nonlinear differential equations (15), the quantity $\bar\lambda$, and the amplitude A of the generalized coordinate $r_1(t)$ as follows:

$$\begin{aligned} C_{11} = \bar\sigma_1^2 - 1 + \bar\lambda^2\Bigg(1 + A^2\bigg(\frac{3}{4}d_1 + d_3\Big(g_0 + \frac{1}{2}g_2\Big) + d_4\Big(y_1 - \frac{1}{2}y_3\Big) \\ + d_5\Big(f_0 + \frac{1}{2}f_2\Big) + d_6\Big(x_1 - \frac{1}{2}x_3\Big)\bigg)\Bigg) - A^2\bar\lambda\bigg(\Big(\frac{1}{2}d_3 - 2d_4\Big)y_4 + \Big(\frac{1}{2}d_5 - 2d_6\Big)x_4\bigg) \\ + A^2\bigg(-\frac{3}{2}d_1 - d_3(g_0 + y_1) - d_5(f_0 + x_1) + \Big(\frac{1}{2}d_3 - 2d_4\Big)(g_2 - y_3) \\ + \Big(\frac{1}{2}d_5 - 2d_6\Big)(f_2 - x_3)\bigg), \end{aligned}$$

$$\begin{aligned} C_{12} = A^2\bar\lambda^2\bigg(d_4\Big(y_2 - \frac{1}{2}y_4\Big) + d_6\Big(x_2 - \frac{1}{2}x_4\Big)\bigg) \\ + \bar\lambda\bigg(2 + A^2\Big(d_1 + 2d_3g_0 + 2d_5f_0 + \Big(\frac{1}{2}d_3 - 2d_4\Big)y_3 + \Big(\frac{d_5}{2} - 2d_6\Big)x_3\Big)\bigg) \\ + A^2\bigg(-d_3y_2 - d_5x_2 - \Big(\frac{1}{2}d_3 - 2d_4\Big)y_4 - \Big(\frac{1}{2}d_5 - 2d_6\Big)x_4\bigg), \end{aligned}$$

$$C_{21} = A^2\bar{\lambda}^2\frac{1}{2}\left(d_4y_4 + d_6x_4\right) - \bar{\lambda}\Bigg(2 + A^2\Bigg(d_1 + d_3\left(2g_0 + y_1\right) + d_5(2f_0 + x_1)$$

$$+ \left(\frac{1}{2}d_3 - 2d_4\right)y_3 + \left(\frac{1}{2}d_5 - 2d_6\right)x_3\Bigg)\Bigg) + A^2\Bigg(\left(\frac{1}{2}d_3 - 2d_4\right)y_4 + \left(\frac{1}{2}d_5 - 2d_6\right)x_4\Bigg),$$

$$C_{22} = \bar{\sigma}_1^2 - 1 + \bar{\lambda}^2\Bigg(1 + A^2\Bigg(\frac{1}{4}d_1 + d_3\left(g_0 - \frac{1}{2}g_2\right) - \frac{1}{2}d_4y_3 + d_5\left(f_0 - \frac{1}{2}f_2\right) - \frac{1}{2}d_6x_3\Bigg)\Bigg)$$

$$+ A^2\bar{\lambda}\Bigg(- d_3y_2 - d_5x_2 - \left(\frac{1}{2}d_3 - 2d_4\right)y_4 - \left(\frac{1}{2}d_5 - 2d_6\right)x_4\Bigg)$$

$$+ A^2\Bigg(- \frac{1}{2}d_1 - d_3g_0 - d_5f_0 - \left(\frac{1}{2}d_3 - 2d_4\right)\left(g_2 + y_3\right) - \left(\frac{1}{2}d_5 - 2d_6\right)(f_2 + x_3)\Bigg), \quad (31)$$

$$x_1 = \frac{-\bar{\lambda}^2d_{14} + 2d_{14} - 2d_8}{2(\bar{\lambda}^2 + \bar{\sigma}_0^2)}, \qquad x_2 = \frac{\bar{\lambda}(d_8 - d_{14})}{\bar{\lambda}^2 + \bar{\sigma}_0^2},$$

$$x_3 = \frac{2d_8(4 + 3\bar{\lambda}^3 - \bar{\sigma}_0^2) + d_{14}(\bar{\lambda}^4 + (\bar{\lambda}^2 - 2)(\bar{\sigma}_0^2 + 2) + 12)}{2(\bar{\lambda}^4 + 2\bar{\lambda}^2(\bar{\sigma}_0^2 + 4) + (\bar{\sigma}_0^2 - 4)^2)}, \quad (32)$$

$$x_4 = \frac{\bar{\lambda}(\bar{\lambda}^2(d_8 - d_{14}) + (d_8 + d_{14})\bar{\sigma}_0^2)}{\bar{\lambda}^4 + 2\bar{\lambda}^2(\bar{\sigma}_0^2 + 4) + (\bar{\sigma}_0^2 - 4)^2},$$

$$y_1 = \frac{-\bar{\lambda}^2d_{15} + 2d_{15} - 2d_7}{2(\bar{\lambda}^2 + \bar{\sigma}_2^2)}, \qquad y_2 = \frac{\bar{\lambda}(d_7 - d_{15})}{\bar{\lambda}^2 + \bar{\sigma}_2^2},$$

$$y_3 = \frac{2d_7(4 + 3\bar{\lambda}^3 - \bar{\sigma}_2^2) + d_{15}(\bar{\lambda}^4 + (\bar{\lambda}^2 - 2)(\bar{\sigma}_2^2 + 2) + 12))}{2(\bar{\lambda}^4 + 2\bar{\lambda}^2(\bar{\sigma}_2^2 + 4) + (\bar{\sigma}_2^2 - 4)^2)}, \quad (33)$$

$$y_4 = \frac{\bar{\lambda}(\bar{\lambda}^2(d_7 - d_{15}) + (d_7 + d_{15})\bar{\sigma}_2^2)}{\bar{\lambda}^4 + 2\bar{\lambda}^2(\bar{\sigma}_2^2 + 4) + (\bar{\sigma}_2^2 - 4)^2}.$$

Since the system of linear algebraic equations (30) with respect to the constants a_1 and b_1 must have a nonzero solution (otherwise, $a_1 = b_1 = 0$), the determinant of this system must be equal to zero, i.e.,

$$\mathbf{D}(\lambda) = \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} = 0. \quad (34)$$

Expanding determinant (34), we obtain a characteristic equation, which, in the general case, is a polynomial of degree 24 in $\bar{\lambda}$. The derivation of this equation in explicit form with regard for relations (31)–(33) is a rather complicated procedure. Therefore, for the investigation of the values of the characteristic exponents λ, it is necessary to find all roots of the characteristic determinant (34) by numerical methods. Thus, unstable motions correspond to the case where there exist characteristic exponents with nonzero real part $(\mathrm{Re}\lambda \neq 0)$. Stable oscillations are associated with imaginary roots $(\mathrm{Re}\lambda = 0)$ of the characteristic determinant (34), which, according to the Hill classification, belong to solutions of the neutral type of the corresponding variational equations.

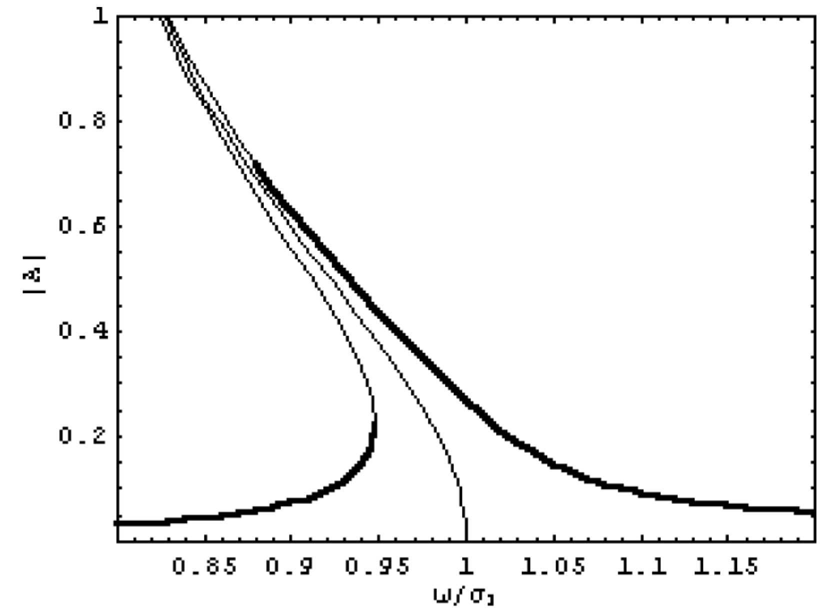

Fig. 2. Stable branches of the amplitude–frequency characteristics of oscillations of a liquid.

In Fig. 2, we present the amplitude–frequency characteristic of oscillations of a liquid with parameters (36) computed by using relations (24). The amplitudes of stable and unstable oscillations are depicted by heavy and less heavy lines, respectively. It follows from the form of the curves that the frequency depends nonlinearly on the amplitude of oscillations. The thin line represents the dependence of the frequency on the amplitude for free oscillations of the liquid.

Thus, we have reduced the problem of investigation of the stability of periodic solutions (24) of system (15), which describe the motion of the liquid in a given cavity, to the problem of determination of the roots of the characteristic determinant (34).

It should also be noted that we can apply the method of slowly varying amplitudes [6] to the problem of finding approximate expressions for the generalized coordinates $p_0(t),\ r_1(t),\ p_2(t),$ and $r_3(t).$ This enables us to trace transient modes, if they exist.

5. Analysis of Amplitude–Frequency Characteristics of Nonlinear Oscillations of the Free Surface of a Liquid

The amplitude–frequency characteristics of nonlinear oscillations of the free surface of a liquid are determined by Eq. (25). For the stationary modes of motion, we can describe the evolution of the free surface of the liquid in each particular case by using the following representation for the free surface:

$$x = p_0(t)Y_0(k_0\xi) - r_1(t)Y_1(k_1\xi)\sin\eta - p_2(t)Y_2(k_2\xi)\cos 2\eta + r_3Y_3(k_3\xi)\cos 3\eta. \qquad (35)$$

In Fig. 3, we display the amplitude–frequency characteristics of oscillations of the liquid calculated according to relations (24) for the cylindrical sector with half-angle $\alpha = \dfrac{\pi}{2}$ with the parameters

$$R_0 = 0, \quad R = 1, \quad d = 2R, \quad h = 2, \quad H = 0.01094; \qquad (36)$$

we also present there the experimental data obtained in [7]. We denote the mean amplitude (which is equal to the

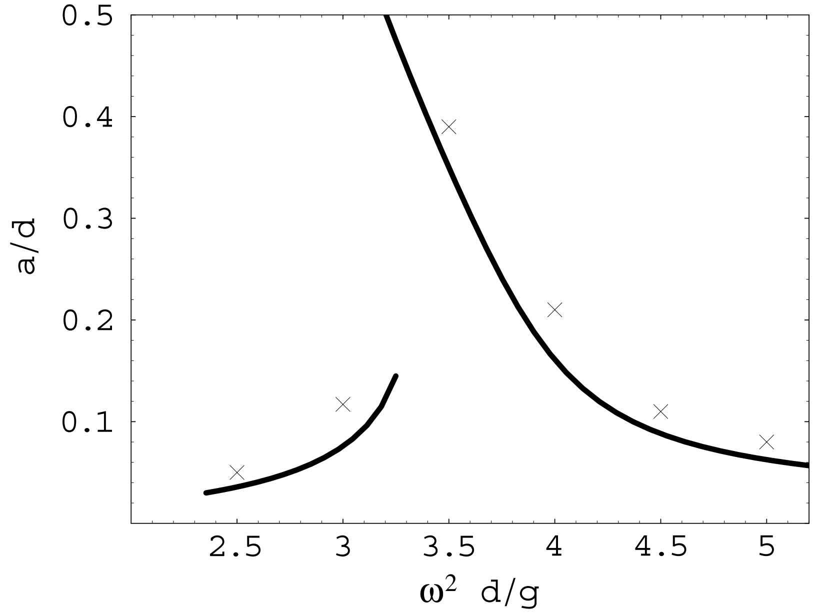

Fig. 3. The value of the mean amplitude.

half sum of the amplitudes of two peaks measured near the wall in the plane of action of the perturbation force) by a and the acceleration of gravity by g. In the case considered, the mean amplitude is determined by the relation

$$a = \frac{d}{2}(|p_0(t) + r_1(t) - p_2(t)| + |p_0(t) - r_1(t) - p_2(t)|). \tag{37}$$

One can see that there the theoretical results agree well with the experimental data marked by the symbol $\times$ in Fig. 3.

The analysis of the evolution of the free surface of the liquid performed for the cylindrical vertical sector with parameters (36) for the profile of wave (35) [constructed by using relations (24) and (25)] and different moments of time t shows that the height of the "hump" is greater than the depth of the "valley." Therefore, the perturbed free surface of the liquid is nonsymmetric. The location of the nodal line varies with time, whereas, according to the linear theory, the nodal line is fixed.

This is especially apparent for the maximum values of amplitudes, which are equal to 0.60476 and 0.29458 in the case considered, i.e., the ratio of the height of the "hump" and the depth of the "valley" is equal to 2.05291 (for $t = 0.78$).

The difference between the height of the "hump" H and the depth of the "valley" increases with the amplitude of perturbations. Note that, in this case, the period of oscillations is equal to $T = 2\pi/\omega = 1.08008$.

Thus, it follows from the examples presented above that, by using the method proposed, one can qualitatively and quantitatively investigate the kinematics of nonlinear oscillations of a liquid near the principal resonance in the tank with barrier. The results obtained are in good agreement with the experimental data presented in [7]. Therefore, taking into account four generalized coordinates, which corresponds to retaining the first four natural forms of oscillations of the free surface of the liquid (two symmetric and two asymmetric) in the expansion of the free surface, one can fairly completely (both qualitatively and quantitatively) describe nonlinear effects caused by the evolution of the free surface.

6. Force Interaction of the Liquid and the Vessel

Consider the problem of force interaction of a liquid with a partially filled tank, which is important in practice. As is known, the principal vector of forces exerted by the liquid on the tank is determined as follows:

$$\vec{P} = \iint_S p\vec{n}dS, \tag{38}$$

where $\vec{n}$ is the unit vector of the outward normal to the wetted surface S and p is the pressure of the liquid determined from the Lagrange–Cauchy integral

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - \nabla\Phi\cdot\dot{\vec{u}} - \vec{g}\cdot\vec{r} + \frac{p}{\rho} = 0. \tag{39}$$

Here, $g = (-g, 0, 0)$, and $\vec{u} = (0, 0, H\cos\omega t)$ gives the law of motion of the tank.

The direct application of relation (38) is extremely difficult in practice. To find an expression for the hydrodynamic force, we use the results presented in Sec. 13 of [3]. As a result, we get

$$\vec{P} = -m(\ddot{\vec{u}} - \vec{g}) - \frac{d\vec{K}}{dt}, \tag{40}$$

where m is the mass of the liquid and $\vec{K}$ is the momentum of the liquid defined by the formula

$$\vec{K} = \rho\iiint_Q (\nabla\Phi)dQ. \tag{41}$$

In the general case where the equation of the perturbed free surface Σ has the form

$$x = \sum_i \beta_i(t)f_i(y,z), \tag{42}$$

we obtain the following relations for the projections of the momentum $\vec{K}$ to the axes of the Cartesian coordinate system:

$$K_x = \frac{1}{2}\sum_i \lambda_{i1}\beta_i(t)\dot{\beta}_i(t),$$
$$K_y = \sum_i \lambda_{i2}\dot{\beta}_i(t), \qquad K_z = \sum_i \lambda_{i3}\dot{\beta}_i(t), \tag{43}$$

where

$$\lambda_{i1} = \rho\int_{\Sigma_0} f_i^2(y,z)dS, \qquad \lambda_{i2} = \rho\int_{\Sigma_0} yf_i(y,z)dS, \qquad \lambda_{i3} = \rho\int_{\Sigma_0} zf_i(y,z)dS. \tag{44}$$

In the case considered, the quantities $p_0(t)$, $r_1(t)$, $p_2(t)$, and $r_3(t)$ play the role of the generalized coordinates $\beta_i(t)$. The projections of the hydrodynamic force to the axes of the coordinate system have the following form up to terms of the third order of smallness:

$$P_x = -mg - \lambda_{21}(r_1\ddot{r}_1 + \dot{r}_1^2),$$
$$P_y = -\lambda_{12}\ddot{p}_0 - \lambda_{32}\ddot{p}_2, \tag{45}$$
$$P_z = -m\ddot{u} - \lambda_{23}\ddot{r}_1,$$

where

$$\lambda_{21} = \frac{\pi}{2}\rho\int_{R_0}^{R}\xi Y_1^2(k_{11}\xi)d\xi, \qquad \lambda_{12} = 2\rho\int_{R_0}^{R}\xi^2 Y_0(k_{01}\xi)d\xi,$$
$$\lambda_{32} = \frac{2}{3}\rho\int_{R_0}^{R}\xi^2 Y_2(k_{21}\xi)d\xi, \qquad \lambda_{23} = \frac{\pi}{2}\rho\int_{R_0}^{R}\xi^2 Y_1(k_{11}\xi)d\xi. \tag{46}$$

For practical purposes, the most important in the case under consideration is the component of the total hydrodynamic force P_z along the Oz-axis (along which the forced oscillations of the sector take place). Substituting the expressions for $u(t)$ and $r_1(t)$ in the last relation of system (45), we obtain the following relation for the determination of the amplitude of the force:

$$|P_z| = \frac{\pi}{2}\rho\omega^2[R^2hH + Aj], \tag{47}$$

where

$$j = \int_{R_0}^{R} Y_1(k_1\xi)\xi^2 d\xi. \tag{48}$$

Using relation (47), we can estimate the contribution to the projection of the total hydrodynamic force P_z made by the inertial forces

$$|P_z^{if}| = \frac{\pi}{2}\rho\omega^2 R^2 hH \tag{49}$$

and by the wave motion of the free surface of the liquid

$$|P_z^{vm}| = \frac{\pi}{2}\rho\omega^2 Aj. \tag{50}$$

For a cylindrical sector with $R = 1$ and $h = 2$ that harmonically oscillates along the Oz-axis with the arm $H = 0.01094$, relative frequency $\omega/\sigma_1 = 0.949$, and the amplitude of the principal generalized coordinate $A = 0.22336$, the contribution of the amplitude P_z^{vm} to the value of P_z is about 84%. This means that, in finding

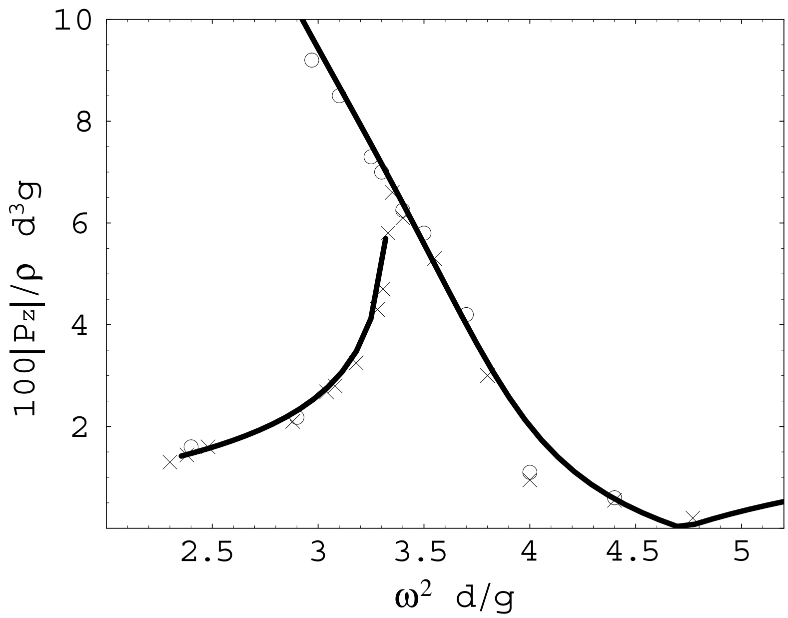

Fig. 4. The amplitude of the force ($H = 0.01094$, $\times$ denotes an increase in frequency, and $\circ$ denotes a decrease in frequency).

the projection of the amplitude of the total hydrodynamic force to the Oz -axis, it is important to determine the amplitude–frequency characteristic $A(\omega/\sigma_1)$ of the generalized coordinate $r_1(t)$ more precisely.

Within the framework of the theory presented, we consider the problem of influence of the vertical diametrical barrier on the character of the force action of the liquid on the tank. It is known [7] that, in the tank halved by a diametrical barrier, the spatial motion in the form of a circular wave is eliminated.

In Fig. 4, we present the amplitude–frequency characteristics of the dimensionless force P_z for a cylinder halved by a vertical barrier with parameters (36); the symbols $\times$ and $\circ$ depict some experimental data from [7]. As in the case of a cylindrical tank without barrier [3], the theoretical and experimental results agree with an accuracy up to $1.5 - 2\%$. Furthermore, in the case where the cylindrical tank is divided into two parts, the maximum value of the projection of the amplitude of the hydrodynamic force to the Oz -axis decreases approximately by 23%.

It should also be noted that the results obtained on the basis of mathematical model (15) agree with experimental data better than the results calculated according to the Hatton model [7, 12] based on the theory of perturbations.

Comparing Figs. 3 and 4, we conclude that, for the projection of the total hydrodynamic force (Fig. 4), the theoretical results agree with experimental data better than for the mean amplitude (Fig. 3).

CONCLUSIONS

In the present paper, we have considered nonlinear oscillations of an ideal incompressible liquid. Using the Bubnov–Galerkin method, we have constructed periodic solutions for the considered four-mode system describing nonlinear forced oscillations of the liquid in a semicircular cylindrical tank in the case of a perturbation force acting along a barrier. We have constructed and investigated the domains of stability and instability of forced oscillations. It has been confirmed that, in the neighborhood of the principal resonance of the system, the amplitudes of forced oscillations of the liquid and the amplitudes of the force are bounded, the location of the nodal line of the free surface of the liquid varies with time, the height of the “hump” of the deformed surface is greater than the depth of the “valley,” and some other nonlinear effects take place. We have established that the presence of a barrier improves the stability of this system (in addition, a partition of this type eliminates spatial motions in the form of a circular wave). We have analyzed the hydrodynamic interaction of the liquid with the tank. The results of computation are in good agreement with experimental data. The results obtained can be used for the design of means of transport containing a large amount of a liquid.

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