Symmetry Breaking and AC-Driven Dynamics of Solitons

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We report on the possibility of unidirectional motion of a topological soliton in a dissipative AC-driven sine-Gordon equation. This motion is due to breaking of certain space-time symmetries. It can be achieved by the proper choice of the potential or the AC force. We formulate the necessary symmetry conditions for the AC force and confirm them with the first order perturbation theory and numerical simulations. The same symmetry considerations are applied to the discrete sine-Gordon equation.

1 Introduction

Transport phenomena based on nonlinear effects are at the heart of many problems in physics. In this context the problem of creation of a net DC current from signals of zero mean is of particular interest [1]. Historically this problem has been posed by Smolouchowski and Feynmann (see [2]) under the name of the *ratchet* effect. It can be described as a Brownian particle in an asymmetric periodic potential, moving in a specific direction in presence of damping under the action of AC forces of zero average. The origin of a net motion is associated with the breaking of space-temporal symmetries of the system [3, 4]. The effect, originally studied for Brownian particles, was generalized to soliton systems with asymmetric potentials [5]. On the other hand, it is known that for a particle an *unidirectional* transport is possible also in symmetric periodic potentials, provided the driving force breaks certain *temporal* symmetries which relate orbits of opposite velocities in phase space [3]. Since in applications it is easier to act on the temporal part (by using external forces) than on the spatial part (by inducing distortion of the potential) of the system, we aim to explore the effect of the temporal symmetry breaking on soliton dynamics.

2 Sine-Gordon equation

Solitons are nonlinear waves that are spatially localized, move with constant velocity and collide elastically. Among the equations that have soliton solutions, the sine-Gordon (SG) equation

$$u_{tt} - u_{xx} = -\sin u, \tag{1}$$

takes a special place. This equation has a number of applications in condensed matter physics, quantum field theory and other areas of modern science. It has soliton solutions in the form of kinks and antikinks: $u(x,t) = 4\tan^{-1}\left[\exp\left(\pm(x-vt)/\sqrt{1-v^2}\right)\right]$. The sign "+" stands for the kink while "−" stands for the antikink. Kinks and antikinks come as a one-parametric family of solutions with the velocity v as a parameter. The velocity spectrum for kinks and antikinks is symmetric with respect to zero: $-1 < v < 1$. In the unperturbed version the solitons of the

SG equation have well-defined particle properties with their mass, energy, momentum

$$P = -\int_{-\infty}^{+\infty} u_x u_t dx, \tag{2}$$

and topological charge

$$Q = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u_x dx. \tag{3}$$

For kinks $Q = 1$ while for antikinks $Q = -1$. For brevity, further on we will consider only kinks. If a weak perturbation is added the particle properties of the topological soliton will survive. The dynamics of the soliton can be approximately described by the dynamics of its center of mass. Thus, one could define the soliton velocity as

$$v = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x u_{xt} dx, \tag{4}$$

bearing in mind that it works only if the soliton can be approximated as a particle, e.g. when the perturbation is weak enough.

We are interested in the AC-driven and damped sine-Gordon (SG) equation

$$u_{tt} - u_{xx} + \sin u = -\alpha u_t + E(t) \equiv \epsilon f(t), \qquad \langle E(t)\rangle_t = 0, \tag{5}$$

where $E(t) = E(t+T)$ is a periodic field with zero mean and α stands for the damping coefficient. For our problem the boundary conditions can be either periodic: $u(x + L, t) = u(x, t) + 2\pi$, $u_x(x + L, t) = u_x(x, t)$, or free ends: $u_x(0, t) = u_x(L, t) = 0$. If the size of the system L is much larger then the size of the soliton, for our purposes both cases should yield the same result. For practical tasks, mostly connected with soliton dynamics in Josephson junctions, the choice of boundary conditions means different physical situations, but this is out of scope of this paper.

Now our aim is to formulate general conditions on the driving field $E(t)$ that will provide directed motion of a soliton. All symmetry operations (see for details [6, 7]) that can relate a soliton solution $u(x, t; v)$ having velocity v with the identical solution but with velocity $-v$

$$\hat{\boldsymbol{S}} u(x, t; v) = u(x, t; -v),$$

must be destroyed. These symmetries can be combinations of all possible shifts and reflections in t, x and u. We also should keep in mind that these symmetries should not change the sign of the topological charge (3) in order to avoid transformations from a kink to an antikink (or vice versa). For the unperturbed SG equation (1) there are only two such operations:

$$\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{X}}}: \ x \to -x, \quad t \to t + T/2, \quad u \to -u + 2\pi, \qquad \hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{T}}}: \ t \to -t + 2t_0.$$

The parameter t_0 is defined by the shape of the function $E(t)$. If the perturbation is added as written in equation (5), the time-reversal symmetry $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{T}}}$ is destroyed due to damping. Thus, the only symmetry left is the $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{X}}}$ symmetry. The SG equation will be invariant under this symmetry if the driving fields satisfy the following condition:

$$E(t + T/2) = -E(t). \tag{6}$$

It is easy to see that if $E(t) = E(t + T)$ and $\langle E(t)\rangle_t = 0$, the only way to change the sign of the function $E(t)$ is to shift it by $T/2$. In the *underdamped* limit $\alpha = 0$ the equation of motion is also invariant under the time reversal symmetry $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{T}}}$ if

$$E(t + t_0) = E(-t + t_0). \tag{7}$$

Thus, in order to achieve directed soliton propagation, one must apply the driving field $E(t)$ that violates the condition of equation (6) in the general dissipative case ($\alpha \neq 0$) and in the non-dissipative case ($\alpha = 0$) both (6) and (7) should be violated.

It has been shown in [8] that equation (5) with the single harmonic drive $[E(t) = E_0 \cos(\omega t + \theta_0)]$ does not yield propagating kinks if the the driving amplitude is small enough. Kinks can propagate either if damping is very weak (see [8]) or when the driving amplitude exceeds certain threshold and the driving frequency is close to the resonance with the phonons $\omega \sim 1$ (see [9]). But in both cases this motion is not *directed*: kinks can move in either direction and proper averaging with respect to the initial phase will yield zero average kink velocity. For our problem we choose the driving field in the following form:

$$E(t) = E_1 \cos \omega t + E_2 \cos (m\omega t + \theta). \tag{8}$$

For this choice condition (6) is violated for any θ if $E_1, E_2 \neq 0$ and m is even, and is not violated for any E_1, E_2 and θ if m is odd. Also, condition (7) is violated if $\theta \neq 0, \pm\pi$ and $E_1, E_2 \neq 0$.

2.1 Perturbation theory

In order to get an idea how the driving field (8) affects the soliton dynamics, we use the soliton perturbation theory (see [10, 8] for details). The first order of the perturbation theory takes an advantage of the particle properties of the SG kink. It is assumed that perturbation $\epsilon f(t)$ of equation (5) is weak, so that the kink shape is not changed under the perturbation and its parameters change in time adiabatically. Therefore the initial problem can be mapped into the dynamics of the kink mass center coordinates $(X(t), v(t) \equiv \dot{X}(t))$, that can be introduced as $u(x,t) = 4\tan^{-1}(\exp\{[x - X(t)]/[1 - \dot{X}^2(t)]^{1/2}\})$. An ordinary differential equation for $X(t)$ can be obtained by differentiating the momentum (2) with respect to time, and using equation (5):

$$\dot{P} = -\alpha P + 2\pi E(t). \tag{9}$$

Assuming $P(t) = 8v(t)/\sqrt{1 - v(t)^2}$, equation (9) can be solved for $P(t)$, and eventually analytical expression of the average kink velocity $\langle v \rangle$, valid in the limit $E_j/\sqrt{\alpha^2 + (j\omega)^2} \ll 1$, $j = 1, 2$, can be obtained (see [7] for details) for $m = 2$:

$$\langle v \rangle = \frac{1}{T}\int_0^T v(\tau)d\tau \simeq \frac{3\pi^3 E_1^2 E_2 \sin(\theta - \theta_0)}{512(\alpha^2 + \omega^2)\sqrt{\alpha^2 + 4\omega^2}}, \qquad \tan\theta_0 = \frac{\alpha(\alpha^2 + 3\omega^2)}{2\omega^3}. \tag{10}$$

Similarly, for $m = 3$ we obtain that $\langle v \rangle = 0$ independently on θ and for an arbitrary initial phase. These results can be easily understood from the symmetry properties of $E(t)$. In the underdamped limit $\alpha \to 0$ we obtain $\theta_0 \to 0$, and, consequently $\langle v \rangle \sim \sin\theta$. Thus, directed soliton motion ceases to exist if $\theta = 0, \pm\pi$. This is not surprising since for these values of θ the driving field equals $E(t) = E_1 \cos(\omega t) \pm E_2 \cos(2\omega t)$ and the symmetry $\hat{\boldsymbol{S}}_{\boldsymbol{T}}$ is restored because $E(t) = E(-t)$.

2.2 Numerical simulations

In order to check the above considerations we have performed numerical simulations of the perturbed SG equation (5). In Fig. 1 the dynamics of a SG kink, initially at rest, driven by a bi-harmonic driver with $m = 2$ and by a single harmonic driver (i.e. $E_2 = 0$), are reported in Figs. 1(a,b), respectively (note that we use contour plots to show the time evolution surface generated by the kink profile). From these figures we see that, in the case $m = 2$ the soliton center of mass moves with a constant velocity, while for the single harmonic driver, it oscillates around the initial position. Numerical results of [7] show that an antikink with parameters as in Fig. 1a will move in the opposite direction.

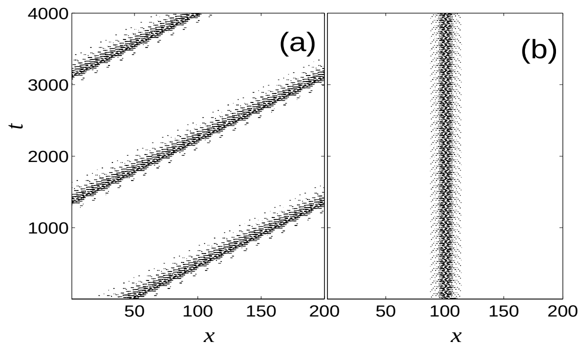

Figure 1. Contour plots of the velocity field $u_x(x,t)$ for (a): $E_1 = 0.4$, $E_2 = 0.26$ and (b): $E_1 = 0.66$, $E_2 = 0$ under periodic boundary conditions. Other parameters are $\alpha = 0.12$, $\omega = 0.25$, $\theta = 1.61$.

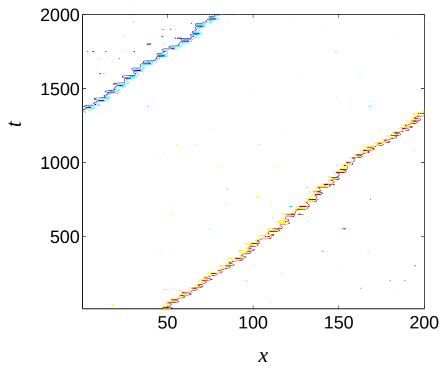

Figure 2. Contour plot for the displacement field $u(x,t)$ in the presence of white noise for parameters as in Fig. 1a.

Let us now briefly investigate the influence of a white noise on the kink dynamics. In order to accomplish this, we have simulated equation (5) with an additional stochastic term $\xi(x,t)$ which satisfies $\langle\xi(x,t)\rangle = 0$, $\langle\xi(x,t)\xi_m(x',t')\rangle = 2\alpha D\delta(x-x')\delta(t-t')$. In Fig. 2[1] we report the contour plot for the kink motion in the case of a bi-harmonic driver with $m = 2$. We see that the noise introduces disturbances on the profile but does not destroy the directed motion of the soliton. We checked that this property remains true also if we increase the amplitude of the noise up to the value when kink-antikink pairs start to nucleate. Moreover, the existence of the phenomena in presence of noise shows the validity of the above mechanism also for the non-deterministic soliton ratchets. Since the simulations with white noise can also be considered as "averaging" over the phase space, our simulations also have demonstrated that there is only one attractor of the system and the computed soliton velocity indeed can be considered and an average one. Similar simulations for $m = 3$ (see [7] for details) show that no directed motion of a kink is possible. This is because for $m = 3$ the $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{X}}}$ symmetry is not broken.

In Fig. 3 we show how the average kink velocity changes as a function of the phase difference between the driving fields, θ. Numerical results clearly confirm analytical approximation of the previous subsection (equation (10)). Another important observation is confirmation of the symmetry argument. As long as the underdamped limit is approached, the curve $\langle v\rangle(\theta)$ crosses the line $\langle v\rangle = 0$ closer and closer to the points $\theta = 0, \pm\pi$. This is in accordance with the fact that the symmetry $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{T}}}$ is restored in this limit and (7) is no longer violated.

3 Discrete sine-Gordon equation

Discrete sine-Gordon (DSG) equation is a modification of its continuous counterpart for the media with spatial discreteness. It is used to model phenomena on lattices, for example fluxon

[1]Figures in colour will be available only in electronic version.

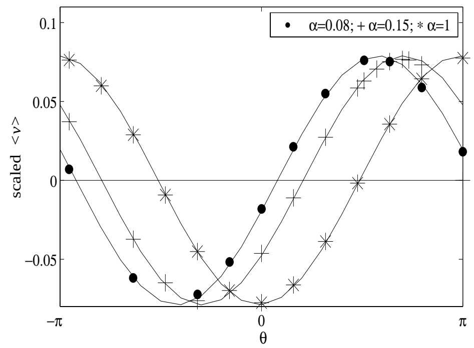

Figure 3. Scaled average kink velocity as a function of the phase difference θ for $E_1 = 0.3$, $E_2 = 0.195$, $\omega = 0.25$; $\alpha = 0.08$ (•), $\alpha = 0.15$ (+) and $\alpha = 1$ (∗). Solid lines are the best fit to the $A\sin(\theta + \theta_0)$ function.

dynamics in parallel Josephson junction arrays, dislocation motion in crystals, etc. The driven and damped DSG has the following form:

$$\ddot{u}_n + \alpha\dot{u}_n - \kappa(u_{n+1} - 2u_n + u_{n-1}) + \sin u_n + E(t) = 0, \qquad n = 1, 2, \ldots, N, \tag{11}$$

Here all the terms and parameters have the same meaning as in the SG counterpart. The parameter κ measures the discreteness of the system, in the limit $\kappa \to \infty$ the above equation reaches the continuum limit. One must mention that DSG is not integrable as compared to the continuous one and there is certain principal difference in its dynamics: kinks in this model cannot propagate with constant velocity (in the unperturbed limit, $\alpha = 0$, $E(t) = 0$). Kinks in DSG are pinned by the lattice (see [11] for details). Similarly to the case of the continuous sine-Gordon, two cases of boundary conditions can be considered: periodic $u_{n+N} = u_n + 2\pi$, $n = 1, 2, \ldots, N$ or free ends.

Discrete kink dynamics can be mapped (with a certain degree of approximation) into dynamics of a single particle. As shown by Braun and Kivshar in [11] this dynamics is described by the dynamics of the kink center $X(t)$: $u_n(t) = 4\arctan\left[\exp[n - X(t)]\right]$. Using this ansatz, the effective equation of motion for the kink center can be obtained:

$$\ddot{X} + \alpha\dot{X} + V'_{\mathrm{PN}}(X) + \tilde{E}(t) = 0, \qquad \tilde{E}(t) \sim E(t). \tag{12}$$

Here the Peierls–Nabarro (PN) potential $V_{\mathrm{PN}}(X) = V_{\mathrm{PN}}(X+1)$, $V'_{\mathrm{PN}}(X) \sim \sin 2\pi X$, takes into account the discreteness of the lattice. Progressive kink motion is analogous to rotations of an AC-driven and damped pendulum. It will be locked to the frequency of the drive and will correspond to limit cycles of equation (12). The average kink velocity thus can be expressed as:

$$\langle v\rangle = \langle \dot{X}(t)\rangle = (m/n)(\omega/2\pi), \qquad m, n \in \mathbb{Z}.$$

Here m corresponds to the number of sites which kink has traveled during the time $nT = 2\pi n/\omega$.

Similarly to the previous case of the continuous SG, in order to achieve directed kink motion, we should destroy all symmetries which relate solutions with opposite velocities. We can do this analysis on the basis of equation (12) and this brings us to the problem, solved some time ago (see papers [3,4]). Operations which change the sign of the velocity are either shift in space and reflection in time or shift in time and reflection is space. In the underdamped case $\alpha = 0$ there are two such symmetries

$$\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{X}}}:\ t \to t + T/2, \quad X \to -X, \qquad \hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{T}}}:\ t \to -t + 2t_0.$$

Situation here is very similar to the continuous case. Equation (12) is invariant under the operation $\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{X}}}$ always only if $E(t) = -E(t + T/2)$ is satisfied (along with $V_{\mathrm{PN}}(-X) = V_{\mathrm{PN}}(X)$

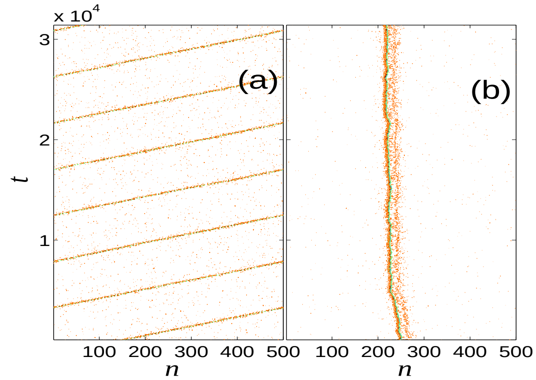

Figure 4. Numerical integration of equation (11) for (a) $E_1 = 0.3$, $E_2 = 0.15$ and (b) $E_1 = 0.45$, $E_2 = 0$. Other parameters are $\kappa = 1$, $\alpha = 0.05$, $\omega = 0.1$, $\theta = 2$. Periodic boundary conditions have been applied.

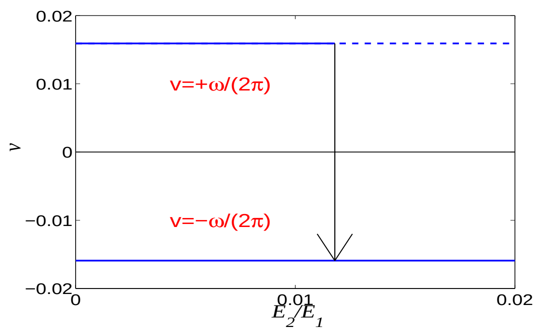

Figure 5. Velocities of two kinks, running into the opposite directions at $\alpha = 0.05$, $\omega = 0.1$, $E_1 = 0.21$, $\theta = 0$ as a function of E_2/E_1. Dashed line denotes that the given solution is no longer stable.

which is always satisfied because the sine-Gordon potential is symmetric). In the non-damping case is also invariant under $\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{T}}}$ if $E(t + t_0) = E(-t + t_0)$. A new symmetry appears in the overdamped limit ($\alpha \to \infty$). In that case the equation of motion can be rewritten as $\dot{X} + V'_{\rm PN}(X) + \tilde{E}(t) = 0$. The symmetry can be written as follows:

$$\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{I}}}: \; t \to -t + t_0, \quad X \to X + 1/2, \tag{13}$$

and equation (12) is invariant under this symmetry if

$$E(t + t_0) = -E(-t + t_0), \qquad V'_{\rm PN}(X) = -V'_{\rm PN}(X + 1/2). \tag{14}$$

If $V'_{\rm PN}(X) \sim \sin 2\pi X$ the second condition is satisfied. Thus, in the overdamped case both (6) and (14) must be violated if we want to obtain directed kink motion.

3.1 Numerical simulations

Verification of the above considerations has been performed via numerical integration of the DSG equation (11) in the presence of weak noise (similarly as in the previous section for the continuous SG) by adding an stochastic term $\xi_n(t)$ $[\langle \xi_n(t) \rangle = 0$, $\langle \xi_n(t)\xi_m(t') \rangle = 2\alpha D \delta_{mn} \delta(t - t')]$ to each of equations (11). Results of simulations are shown in Fig. 4. We clearly observe directed motion of a kink in the case of two mixed harmonics and absence of such motion when the driving field $E(t)$ consists from only one harmonic. Motion of an antikink for the same parameters occurs in the opposite direction with the same absolute value of the average velocity.

Emergence of the directed kink motion is caused by desymmetrization of basins of attraction of two limit cycles with opposite velocities. For one-particle ratchets it has been shown in [3,12].

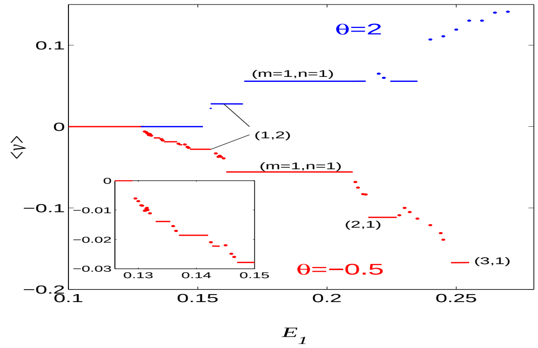

Figure 6. Kink velocity as a function of the driving amplitude $E_1 = E_2$ for $\theta = 2$ and $\theta = -0.5$. Other parameters are: $\omega = 0.35$, $\alpha = 0.1$, $\kappa = 1$. The inset shows more detailed picture for the $\theta = -0.5$ case.

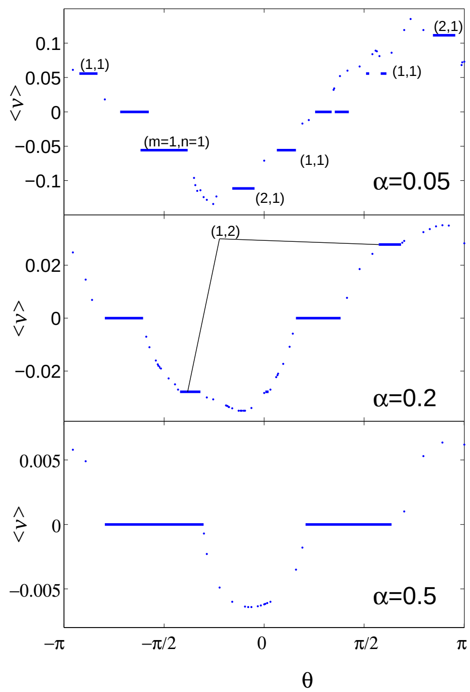

Figure 7. Average kink velocity as a function of the phase θ at $\omega = 0.35$, $E_1 = E_2 = 0.2$, and different values of damping: $\alpha = 0.05$ (upper panel),$\alpha = 0.2$ (middle panel) and $\alpha = 0.5$ (bottom panel).

As long as symmetry is broken (by switching to E_2), one of the basins of attraction begins to shrink and eventually disappears. Thus, only limit cycles which correspond to motion in one direction survive, as shown in Fig. 5. We see that already for rather weak desymmetrizations only one attractor survives.

In Fig. 6 we plot the kink velocity as a function of the driving amplitude. We observe certain differences compared to the case of continuous SG (see also [13] for details on damped DSG): a depinning threshold exists, while for the continuous case $\langle v \rangle \sim E_1^2 E_2$, e.g. kink can move for an arbitrarily small applied field. Since kink dynamics is locked to the driving frequency ω, the steps in the figure correspond to the rational multipliers of $\omega/(2\pi)$ and the dependence has a form of "devils staircases". After certain critical threshold $E_1 = E_2 \sim 0.3$ is passed, dynamics becomes chaotic and there is no kink solutions.

In Fig. 7 we plot average kink velocity as a function of the phase θ. Note that dependence of the kink velocity on the system parameters is no longer smooth. This is the result that dynamics is locked to the time-periodic driving field. Thus, the function $\langle v \rangle(\theta)$ only slightly resembles the sine function, especially for strong dampings when depinning of the kink requires very strong fields. In the underdamped limit $\alpha \to 0$ directed motion disappears approximately at $\theta = 0, \pm\pi$, when $\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{T}}}$ is restored: $E(t) = E(-t)$. Note that the same happens in the continuum case. In the overdamped limit $\alpha \to \infty$ directed motion disappears approximately at $\theta = \pm\pi/2$, when the symmetry $\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{I}}}$ is restored: $E(t) = E_1 \cos\omega t + E_2 \cos(2\omega t + \pi/2) = -E(\pi/\omega - t)$.

4 Conclusions

We have considered a way to produce a directed motion of a topological soliton (in both discrete and continuous media) in the presence of damping, using an appropriate AC driving force of zero mean. The effect has been demonstrated for both continuous and discrete SG equations, but can be extended to other kink-bearing models. We have shown that in order to get directed soliton propagation it is necessary to break certain symmetries of the driving field $E(t)$ and have confirmed our findings both analytically and numerically. The phenomenon can be observed experimentally in an AC biased annular long Josephson junction or annular 1D array of inductively coupled parallel small Josephson junctions.

[1] Reimann P., Brownian motors: noisy transport far from equilibrium, *Phys. Rep.*, 2002, V.361, N 2–4, 57–265.

[2] Smoluchowski M., Experimentell nachweisbare der übrichen Thermodynamik widersprechende Molekularphänomene, *Phys. Z.*, 1912, V.13, 1069–1080;
Feynman R.P., Leighton R.B. and Sands M., The Feynman lectures on physics, Reading MA, Addison-Wesley, 1966.

[3] Flach S., Yevtushenko O. and Zolotaryuk Y., Directed current due to broken time-space symmetry, *Phys. Rev. Lett.*, 2000, V.84, N 11, 2358–2361.

[4] Yevtushenko O., Flach S., Zolotaryuk Y. and Ovchnnikov A.A., Rectification of current in AC-driven nonlinear systems and symmetry properties of the Boltzmann equation, *Europhys. Lett.*, 2001, V.54, N 2, 141–147.

[5] Marchesoni F., Thermal ratchets in $1 + 1$ dimensions, *Phys. Rev. Lett.*, 1996, V.77, N 11, 2364–2367;
Savin A.V., Tsironis G.P. and Zolotaryuk A.V., Ratchet and switching effects in stochastic kink dynamics, *Phys. Lett. A*, 1997, V.229, N 5, 279–284;
Falo F., Martinez P.J., Mazo J.J. and Cilla S., Ratchet potential for fluxons in Josephson-junction arrays, *Europhys. Lett.*, 1999, V.45, N 6, 700–706;
Salerno M. and Quintero N.R, Soliton ratchets, *Phys. Rev. E*, 2002, V.65, N 2, 025602(R).

[6] Flach S., Zolotaryuk Y., Miroshnichenko A.E. and Fistul M.V., Broken symmetries and directed collective energy transport in spatially extended systems, *Phys. Rev. Lett.*, 2002, V.88, N 18, 184101.

[7] Salerno M. and Zolotaryuk Y., Soliton ratchetlike dynamics by AC forces with harmonic mixing, *Phys. Rev. E*, 2002, V.65, N 5, 056603.

[8] Olsen O.H. and Samuelsen M.R., Sine-Gordon 2-Pi-kink dynamics in the presence of small perturbations, *Phys. Rev. B*, 1983, V.28, N 1, 210–217.

[9] Ustinov A.V. and Malomed B.A., Observation of progressive motion of AC-driven solitons, *Phys. Rev. B*, 2001, V.64, N 2, 020302(R).

[10] McLaughlin D.W. and Scott A.C., Perturbation analysis of fluxon dynamics, *Phys. Rev. A*, 1978, V.18, N 4, 1652–1680.

[11] Braun O.M. and Y.S. Kivshar, Nonlinear dynamics of the Frenkel–Kontorova model, *Phys. Rep.*, 1998, V.306, N 1–2, 2–108.

[12] Barbi M. and Salerno M., Phase locking effect and current reversals in deterministic underdamped ratchets, *Phys. Rev. E*, 2000, V.62, N 2, 1988–1994.

[13] Floría L.M. and Mazo J.J., Dissipative dynamics of the Frenkel–Kontorova model, *Adv. Phys.*, 1996, V.45, 505–598.