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Toda Chain, Sheffer Class of Orthogonal Polynomials and Combinatorial Numbers

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A classification of Hankel determinant solutions of the restricted Toda chain equations is presented through polynomial Ansatz for moments. Each solution corresponds to the Sheffer class orthogonal polynomials. In turn, these solutions are equivalent to solutions with separated variables in Toda chain. These solutions lead naturally to explicit Hankel determinants of some combinatorial numbers.

1 Introduction

The Toda chain equations [17]

$$\dot{u}_n = u_n(b_n - b_{n-1}), \quad n = 1, 2, \ldots, \qquad \dot{b}_n = u_{n+1} - u_n, \quad n = 0, 1, \ldots \tag{1}$$

with additional condition

$$u_0 = 0 \tag{2}$$

has the well-known relation with the theory of orthogonal polynomials, where the dot indicates the differentiation with respect to t. In what follows we will call equations (1) with restriction (2) *the restricted Toda chain* (TC) equations.

Let $P_n(x;t)$ be orthogonal polynomials satisfying the three-term recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x), \qquad n = 1, 2, \ldots \tag{3}$$

with initial conditions

$$P_0 = 1, \qquad P_1(x) = x - b_0. \tag{4}$$

We will assume that $u_n \neq 0$, $n = 1, 2, \ldots$. Then, by the Favard theorem [4], there exists a nondegenerate linear functional σ such that the polynomials $P_n(x)$ are orthogonal with respect to it:

$$\sigma(P_n(x)P_m(x)) = h_n \delta_{nm}, \tag{5}$$

where h_n are normalization constants. The linear functional σ can be defined through its moments

$$c_n = \sigma(x^n), \qquad n = 0, 1, \ldots. \tag{6}$$

It is usually assumed that $c_0 = 1$ (standard normalization condition), but we will not assume this condition in the followings. So we will assume that c_0 is an arbitrary nonzero parameter.

Introduce the Hankel determinants

$$D_n = \det(c_{i+j})_{i,j=0,\dots,n-1}, \qquad D_0 = 1, \qquad D_1 = c_0. \tag{7}$$

Then the polynomials $P_n(x)$ can be uniquely represented as [4]

$$P_n(x) = \frac{1}{D_n}\begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}. \tag{8}$$

The normalization constants are expressed as

$$h_n = \frac{D_{n+1}}{D_n}, \qquad h_0 = D_1 = c_0. \tag{9}$$

While the recurrence coefficients u_n satisfy the relation

$$u_n = \frac{h_n}{h_{n-1}} = \frac{D_{n-1}D_{n+1}}{D_n^2}. \tag{10}$$

Thus we have

$$h_n = c_0 u_1 u_2 \cdots u_n. \tag{11}$$

Then we have

Theorem 1. *The following statements are equivalent.*

(i) *The recurrence coefficients u_n, b_n satisfy the TC equations* (1) *with the restriction $u_0 = 0$ (i.e. $\dot{b}_0 = u_1$).*

(ii) *The corresponding orthogonal polynomials $P_n(x;t)$ satisfy the relation*

$$\dot{P}_n(x;t) = -u_n P_{n-1}(x;t). \tag{12}$$

(iii) *The moments c_n satisfy the relation*

$$\dot{c}_n = c_{n+1} + \frac{\dot{c}_0 - c_1}{c_0} c_n, \tag{13}$$

where $c_0(t)$ is an arbitrary differentiable function of t.

See [1, 13], for the proof of this theorem.

We note only that it is commonly assumed that $c_0(t) \equiv 1$, but in what follows we will choose another normalization condition

$$\dot{c}_0 = c_1. \tag{14}$$

Then the condition (13) becomes very simple

$$\dot{c}_n = c_{n+1}, \tag{15}$$

or, equivalently,

$$c_n(t) = \frac{d^n c_0(t)}{dt^n}. \tag{16}$$

Hence, for the Toda chain case, the Hankel determinants $D_n = D_n(t)$ have the form

$$D_n = \det(c_0^{(i+k)})_{i,k=0,\dots,n-1}, \qquad D_0 = 1, \qquad D_1 = c_0, \tag{17}$$

where $c_0^{(j)}$ means the j-th derivative of $c_0(t)$ with respect to t.

Now we have

Proposition 1. *The restricted TC equations are equivalent also to the equations*

$$\frac{d^2 \log D_n}{dt^2} = \frac{D_{n-1}D_{n+1}}{D_n^2}, \qquad n = 1, 2, \ldots. \tag{18}$$

Proof of this proposition is almost obvious. Equations (18) are equivalent to the Hirota bilinear form [6] for the restricted TC equations which was analyzed by many authors (see, e.g. [2, 11]). The parametric determinants such as $D_n(t)$ have played a very fundamental role in the Hirota–Sato theory of integrable dynamical systems as tau-functions. As was noticed in [15,18] the relation (18) for the Hankel determinants of type (17) with (15) was firstly obtained by Sylvester and is known today as the Sylvester theorem [9].

Note also that for the Hankel determinants of the form (17) we have two useful relations

$$b_n = \frac{\dot{D}_{n+1}}{D_{n+1}} - \frac{\dot{D}_n}{D_n} \tag{19}$$

and

$$\dot{h}_n = h_n b_n. \tag{20}$$

In particular, for $n = 0$ we have from (20)

$$b_0 = \frac{\dot{c}_0}{c_0}. \tag{21}$$

The relation (21) allows us to restore $c_0(t)$ if the recurrence coefficient $b_0 = b_0(t)$ is known explicitly from Toda chain solutions (1).

In this paper we consider a class of explicit solutions of the restricted TC equations through a separation of variables $c_n(t)$. Such solutions correspond to the Sheffer class orthogonal polynomials such as the Meixner, Pollaczek, Laguerre, Charlier and Hermite polynomials. It is shown that Hankel determinants of some combinatorial numbers, such as the Euler, the binomial coefficients and Bell numbers, are then presented.

2 Generating functions of the moments

In the theory of orthogonal polynomials the Stieltjes function $F(z)$ is defined as a generating function of the moments [4]

$$F(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_n}{z^{n+1}} + \cdots. \tag{22}$$

If moments c_n depend on t according to the Toda Ansatz (15), we then have

$$\dot{F}(z;t) = \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_n}{z^n} + \cdots = zF(z) - c_0. \tag{23}$$

In fact, the relation (23) is equivalent to restricted TC equations (15).

We consider also a generating function of another type:

$$\Phi(p) = \sum_{k=0}^{\infty} c_k \frac{p^k}{k!}. \tag{24}$$

Note that in number theory for a given sequence of numbers c_n the generating function of type (22) is called a G-function, and the generating function of type (24) is called an E-function [5]. The relationship between functions $F(z)$ and $\Phi(p)$ is known and is given by the (formal) Laplace transform:

$$F(z) = \sum_{k=0}^{\infty} c_k z^{-k-1} = \sum_{k=0}^{\infty} c_k \int_0^{\infty} \frac{p^k e^{-pz}}{k!} dp = \int_0^{\infty} e^{-pz} \Phi(p) dp. \tag{25}$$

For the case of the restricted TC equations with condition (15) we see that generating function $\Phi(p)$ is given automatically by the formal Taylor expansion

$$\Phi(p;t) = \sum_{k=0}^{\infty} c_k(t)\frac{p^k}{k!} = \sum_{k=0}^{\infty} c_0^{(k)}(t)\frac{p^k}{k!} = c_0(t+p) \tag{26}$$

of $c_0(t+p)$. Thus the E-generating function is given just by the shifted $c_0(t+p)$ zero-moment function. The Stieltjes function is given then as the Laplace transform

$$F(z;t) = \int_0^{\infty} e^{-pz} c_0(t+p)dp. \tag{27}$$

3 Special polynomial Ansatz and separated variables

In this section we describe concrete examples of the E-generating functions $c_0(t)$ (26) which stem from special Ansatz for functional structure of the moments $c_n(t)$. Namely, we assume a separation of variables as follows

$$c_n(t) = T_n(y(t))c_0(t), \qquad n = 0, 1, \ldots, \tag{28}$$

where $T_n(y(t))$ is a polynomial of exactly degree n of some (unknown) variable $y(t)$. Note that in [7, 8] some systems of orthogonal polynomials were considered having moments as orthogonal polynomials from some variable. In our approach we *do not* require that $T_n(y)$ be orthogonal polynomials.

The main result is

Theorem 2. *In order to Ansatz* (28) *be compatible with the restricted TC equations it is necessary and sufficient that the function* $y(t)$ *be a solution of the equation*

$$\dot{y}(t) = \sigma(y), \tag{29}$$

where $\sigma(y)$ *is a (nonzero) polynomial in* y *with degree less or equal 2,*

$$\sigma(y) = \xi y^2 + \eta y + \zeta, \tag{30}$$

and the function $\phi(y) = c_0(t(y))$ *be a solution of the equation*

$$\phi'(y) = \frac{\tau(y)}{\sigma(y)}\phi(y), \tag{31}$$

where $\tau(y)$ *is a polynomial of exactly first degree,*

$$\tau(y) = \alpha y + \beta \tag{32}$$

with $\alpha \neq 0$ *and* β *arbitrary parameters, and* $t(y)$ *is the inverse function with respect to* $y(t)$. *The restriction between* α *and* ξ, $\xi \neq -\alpha/n$, $n = 1, 2, \ldots$, *is assumed.*

It is possible also to find explicit expression for the recurrence coefficients:

Proposition 2. *If moments* $c_n(t)$ *satisfy the polynomial Ansatz (28) then the recurrence coefficients* $u_n(t)$, $b_n(t)$ *have the explicit expressions*

$$\begin{aligned} b_n(t(y)) &= \tau(y) + n\sigma'(y), \\ u_n(t(y)) &= n\sigma(y)\left(\tau'(y) + \frac{1}{2}(n-1)\sigma''(y)\right) = n\sigma(y)(\alpha + (n-1)\xi). \end{aligned} \tag{33}$$

Proof of this proposition is an elementary application of induction.

We see that the recurrence coefficient $u_n(t)$ has expression with separated variables:

$$u_n(t) = q(t)\kappa_n, \qquad n = 0, 1, \ldots, \tag{34}$$

where $q(t) = \sigma(y(t))$ depends only on t and $\kappa_n = n(\alpha + (n-1)\xi)$ depends only on n.

It is possible to prove an inverse theorem

Theorem 3. *Two Ansatzes* (28) *and* (34) *(with the restriction $\kappa_0 = 0$) for solutions of the restricted TC equations are equivalent.*

Such solutions were constructed in [12] and [19]. They were also rediscovered in [3]. In [19] and [3] it was established, that corresponding polynomials belong to the Sheffer class.

4 Hankel determinants of some combinatorial numbers

In this section we consider some special cases of the general scheme of separated variables. Corresponding Toda solutions lead to simple determinantal formulas containing classical combinatorial numbers.

Case 1. $\sigma(y) = 1 - y^2$. For a special choice of parameters we have $\tau(y) = -y$ and $c_0 = 2e^t/(e^{2t}+1)$. For recurrence coefficients we have

$$b_n(t) = -(1+2n)\tanh(t), \qquad u_n(t) = -n^2\mathrm{sech}^2(t). \tag{35}$$

As discussed in [19, 3] this solution is related to the Meixner orthogonal polynomials.

It is to be noted that the moment $c_0(t)$ has an intimate relationship to the Euler numbers E_k, $k = 0, 1, 2, \ldots$, which are combinatorial numbers defined by

$$E_k = i^k \sum_{m=0}^{k} 2^m a_m \begin{pmatrix} k \\ m \end{pmatrix}, \qquad \frac{2}{e^t+1} = \sum_{k=0}^{\infty} a_m \frac{t^m}{m!}, \qquad i = \sqrt{-1}. \tag{36}$$

Here a_m are rational numbers and $\begin{pmatrix} k \\ m \end{pmatrix}$ are binomial coefficients. Thus $c_0(t) = \Phi(2t; 1/2)$ and the coefficients $E_k(1/2)$ of the expansion

$$c_0(t) = \sum_{k=0}^{\infty} E_k(1/2)\frac{2^k t^k}{k!} \tag{37}$$

give the Euler numbers E_k through the relation $(2i)^k E_k(1/2) = E_k$.

Proposition 3. *The Hankel determinant of the Euler numbers is given by*

$$\det(E_{i+j})_{i,j=0,\ldots,n-1} = \left(\prod_{k=0}^{n-1} k!\right)^2. \tag{38}$$

Radoux [15] presented the Hankel determinant $\det(E_{2i+2j})_{i,j=0,\ldots,n-1} = \left(\prod_{k=0}^{n-1}(2k)!\right)^2$ of nonzero Euler numbers E_{2i+2j}.

Case 2. $\sigma(y) = -(y-1)^2$. In this case $\sigma(y)$ has a real root of multiplicity 2. For a special choice of parameters we have

$$b_n(t) = -z + \frac{\alpha - 2n}{t+1}, \qquad u_n(t) = \frac{n(n-\alpha-1)}{(t+1)^2}. \tag{39}$$

with some parameter z. These coefficients correspond to the Laguerre polynomials. Then we obtain

$$c_0(t) = (t+1)^\alpha e^{-zt+z}. \tag{40}$$

Expression $(t+1)^\alpha e^{-zt}$ is a generating function of the Laguerre polynomials ${L_k}^{\alpha-k}(z)$:

$$(t+1)^\alpha e^{-zt} = \sum_{k=0}^{\infty} {L_k}^{\alpha-k}(z)t^k.$$

From this relation it follows that ${L_k}^{\alpha-k}(0) = \binom{\alpha}{k}$ are (generalized) binomial coefficients.

Let us consider the Hankel determinant of the generating function of the generalized binomial coefficients ${L_k}^{\alpha-k}(0)$

$$\begin{aligned}
&D_n(t) = \det(d_{i+j}(t))_{i,j=0,\ldots,n-1}, \qquad D_0(t) = 1, \qquad D_1(t) = d_0,\\
&d_0(t) = (t+1)^\alpha, \qquad d_n(t) = \frac{d^n d_0(t)}{dt^n}.
\end{aligned} \tag{41}$$

Proposition 4. *The Hankel determinant of the generalized binomial coefficients ${L_k}^{\alpha-k}(0)$ is given by*

$$D_n(0) = (-1)^{[n/2]} \prod_{k=1}^{n-1} k!(\alpha-k+1)^{n-k}. \tag{42}$$

Case 3. $\sigma(y) = y+1$. For special choice of parameters the solution of the restricted TC equations is

$$b_n(t) = u_n(t) = ne^t \tag{43}$$

which corresponds to the Charlier orthogonal polynomials [19].

For $c_0(t)$ we have

$$c_0(t) = \exp\left(e^t - 1\right).$$

This expression coincides with the generating function of the Bell numbers B_k. Namely,

$$c_0(t) = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \qquad B_k = \sum_{m=0}^{k} S(k,m), \tag{44}$$

where $S(k,m)$ is the Stirling numbers of the second kind. Thus $B_k = c_0^{(k)}(t)$. It is known [15] that the associated Hankel determinants

$$D_n(t) = \det(c_0^{(i+j)}(t))_{i,j=0,\ldots,n-1} \qquad \text{and} \qquad D_n(0) = \det(B_{i+j})_{i,j=0,\ldots,n-1}$$

are

$$\begin{aligned}
&D_n(t) = \prod_{k=0}^{n-1} k! \cdot \exp\left(\frac{n(n-1)}{2}t + ne^t - n\right),\\
&D_n(0) = \prod_{k=0}^{n-1} k!,
\end{aligned} \tag{45}$$

respectively.

Case 4. $\sigma(y) = 1$. In this case we have (shifted) Hermite polynomials:

$$b_n(t) = -2t + 2z, \qquad u_n(t) = -2n. \tag{46}$$

The E-generating function

$$c_0(t) = \exp\left(-t^2 + 2zt\right) \tag{47}$$

of the Hermite polynomials $H_k(z)$ is derived, namely,

$$c_0(t) = \sum_{k=0}^{\infty} H_k(z) \frac{t^k}{k!}. \tag{48}$$

The Hankel determinants $D_n = \det(H_{i+j}(z))_{i,j=0,\ldots,n-1}$ of Hermite polynomials is found in Radoux [14] by an alternative way. The result is

$$D_n = \det(H_{i+j}(z))_{i,j=0,\ldots,n-1} = (-2)^{n(n-1)} \prod_{k=0}^{n-1} k!. \tag{49}$$

Note that D_n is independent of z.

Thus the Hankel determinants of some combinatorial numbers can be easily calculated explicitly from corresponding Toda solutions with separated variables.

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