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Symmetries of Radial Maxwell–Vlasov Equations

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Lie group of point symmetries is determined for Vlasov–Maxwell equations of spherically symmetric plasmas. The case of multi-component plasma is considered. Classification of possible invariant solutions is presented. Comparison with the case of one-dimensional rectilinear motion of plasma particles is made.

1 Introduction

The case of purely radial motion of charges is singled out by the general properties of Maxwell equations. Similarly as in one-dimensional rectilinear motion of particles of plasma there is no generation of time dependent magnetic field in this case [1]. Thus, there is no radiation and the self-consistent solutions of the Vlasov–Maxwell equations (distribution functions, electric field and no magnetic field) exist.

The case of one-dimensional rectilinear motion is widely employed in plasma physics [2–5]. It is used as a starting point for analysis of more complicated systems with very significant applications. The same is expected in the case of radial motion, however, this case is not popular and not widely employed. Thus, research into that topic is important and promising.

In this paper we find Lie symmetry groups of point transformations of the Vlasov–Maxwell equations for collision-less multi-component plasma without magnetic field in the case of purely radial motion of particles. We find the optimal system of one-parameter subgroups and classify corresponding invariant solutions.

We use the direct method for determining symmetry groups of integro-differential equations (IDE's) presented in papers [6–8]. For the IDE's of the type

$$F(x_1,\dots,x_n,y,\underset{1}{y},\dots,\underset{m}{y})+\int_X dx_1\cdots dx_l f(x_1,\dots,x_n,y,\underset{1}{y},\dots,\underset{k}{y})=0, \tag{1}$$

where the symbol $\underset{m}{y}$ denotes the set of all partial derivatives of m-order, it consists in the following infinitesimal criterion of invariance

$$G^{(m)}F+\int_X dx_1\cdots dx_l\left[G^{(k)}f+f\sum_{i=1}^{l}\partial_i\xi_i\right]=0 \qquad \text{on solutions of (1)}, \tag{2}$$

where $G^{(m)}$ is the extended to m-th order generator of the point transformation

$$\widetilde{x}^i=e^{\epsilon G}x^i=x^i+\epsilon\xi^i(x,y)+\mathcal{O}(\epsilon^2),$$
$$\widetilde{y}=e^{\epsilon G}y=y+\epsilon\eta(x,y)+\mathcal{O}(\epsilon^2), \tag{3}$$

with the generator

$$G=\xi^i(x,y)\partial_{x^i}+\eta(x,y)\partial_y. \tag{4}$$

The method is a natural generalization of the well known Ovsiannikov's method for partial differential equations which can be found in monographs [9–12]. The short review with relevant references of other methods of finding symmetries of integro-differential equations is given in [8].

2 Symmetries of radial multi-component plasma

In the case of purely radial motion the Vlasov–Maxwell system of equations for collision-less, multi-component, plasmas without magnetic field has the following form

$$
\begin{aligned}
&\partial_t f_\alpha + u\partial_r f_\alpha + \frac{q_\alpha}{m_\alpha} E\partial_u f_\alpha = 0,\\
&\partial_t E + \sum_\alpha \frac{q_\alpha}{\epsilon_0}\int_0^\infty du\, u^3 f_\alpha = 0, \qquad \partial_r E + \frac{2}{r}\,E - \sum_\alpha \frac{q_\alpha}{\epsilon_0}\int_0^\infty du\, u^2 f_\alpha = 0,
\end{aligned} \tag{5}
$$

where $E = E(t,r)$ is the radial component of electric field vector, u is the radial component of vector velocity, $f_\alpha = f_\alpha(t,r,u)$ is the radial distribution function of α-plasma component (integrated over the solid angle in velocity space with 4π included), q_α, m_α are charge and mass of α-particles respectively, and ϵ_0 is electric permittivity of free space.

In this case, the generators (4) of point transformations (3) take the form

$$
G = \tau\partial_t + \xi\partial_r + \rho\partial_u + \sum_\alpha \eta_\alpha\partial_{f_\alpha} + \zeta\partial_E. \tag{6}
$$

Using the criterion (2) we obtain

$$
0 = \partial_{f_\alpha}\tau = \partial_{f_\alpha}\xi = \partial_{f_\alpha}\rho = \partial_E\tau = \partial_E\xi = \partial_E\rho,
$$

and the following determining equations (limits 0 and ∞ of integrals are dropped):

$$
\begin{aligned}
&0 = \partial_E\eta_\alpha \qquad 0 = \partial_{f_\alpha}\zeta, \qquad 0 = u\partial_u\tau - \partial_u\xi, \qquad 0 = \left(\frac{q_\alpha}{m_\alpha} - \frac{q_\beta}{m_\beta}\right)\partial_{f_\beta}\eta_\alpha \quad \text{for } \alpha \neq \beta,\\
&0 = \partial_t\eta_\alpha + u\partial_r\eta_\alpha + \frac{q_\alpha}{m_\alpha}\,E\partial_u\eta_\alpha, \qquad 0 = u\partial_t\tau - \partial_t\xi + \rho + u^2\partial_r\tau - u\partial_r\xi,\\
&0 = \partial_t\zeta + \frac{2}{r}\,E\partial_t\xi, \qquad 0 = \partial_r\zeta - \frac{2}{r^2}\,E\xi + \frac{2}{r}\,\zeta - \frac{2}{r}\,E\partial_E\zeta + \frac{2}{r}\,E\partial_r\xi,\\
&0 = \frac{q_\alpha}{m_\alpha}E\left(\partial_t\tau + u\partial_r\tau + \frac{q_\alpha}{m_\alpha}E\partial_u\tau - \partial_u\rho\right) + \frac{q_\alpha}{m_\alpha}\,\zeta - \partial_t\rho - u\partial_r\rho,\\
&0 = (\partial_t\tau - \partial_E\zeta)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u^3 f_\beta - (\partial_t\xi)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u^2 f_\beta\\
&\qquad + \sum_\beta \frac{q_\beta}{\epsilon_0}\int du(3u^2\rho f_\beta + u^3\eta_\beta + u^3 f_\beta\partial_u\rho),\\
&0 = (\partial_E\zeta - \partial_x\xi)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u^2 f_\beta + (\partial_r\tau)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u^3 f_\beta\\
&\qquad - \sum_\beta \frac{q_\beta}{\epsilon_0}\int du\big(2u\rho f_\beta + u^2\eta_\beta + u^2 f_\beta\partial_u\rho\big).
\end{aligned}
$$

For different species $\alpha \neq \beta$ also $q_\alpha/m_\alpha \neq q_\beta/m_\beta$ for real plasmas (except the special case of deuterium and helium nuclei). This leads to a symmetry group which is completely different from the case of one-component plasma when there is only one value of q/m. This is the reason for the oscillatory like generators for one-component plasma in the case of rectilinear motion of particles to appear, as found by V. Taranov [14]. We easily find that

$$
0 = \partial_u\tau = \partial_u\xi = \partial_t\eta_\alpha = \partial_r\eta_\alpha = \partial_u\eta_\alpha = \partial_E\eta_\alpha, \qquad 0 = \partial_{f_\beta}\eta_\alpha \quad \text{for } \alpha \neq \beta, \qquad \zeta = \lambda_1 E.
$$

Then, the last two integro-differential equations lead to

$$0 = \int du \left[f_\alpha (u^3 \partial_u \tau - \lambda_1 u^3 - u^2 \partial_t \xi + 3u^2 \rho + u^3 \partial_u \rho) + u^3 \eta_\alpha \right],$$
$$0 = \int du \left[f_\alpha (\lambda_1 u^2 - u^2 \partial_r \xi + u^3 \partial_r \tau - 2u\rho - u^2 \partial_u \rho) + u^2 \eta_\alpha \right].$$

We assume that the point transformations (3) are analytic functions of the point $(t, r, u, f_\alpha, \zeta)$. In general, analyticity with respect to the parameter ϵ and infinite differentiability with respect to the point is assumed for Lie groups. However, the latter dependence is in fact also analytic due to a physical interpretation. Expanding $\eta_\alpha(f_\alpha)$ in the Taylor series, using the generalized mean value theorem and well known special stationary solutions of the Vlasov–Maxwell equations (5), which depend only on velocity, we find that the coefficients η_α can depend on f_α only linearly $\eta_\alpha = \lambda_2 f_\alpha$. Thus, we can apply the Lagrange lemma of calculus of variations [13] and obtain *differential* equations for integrands.

Solutions of the determining equations lead to the following three generators

$$G_1 = \partial_t, \qquad G_2 = -t\partial_t - 2r\partial_r - u\partial_u + 5\sum_\alpha f_\alpha \partial_{f_\alpha},$$
$$G_3 = -3t\partial_t - r\partial_r + 2u\partial_u + 5E\partial_E, \tag{7}$$

which span the Lie algebra of the group of point symmetry transformations of the Vlasov–Maxwell equations (5). Non-vanishing commutators between these generators are given by

$$[G_1, G_2] = -G_1, \qquad [G_1, G_3] = -3G_1, \qquad [G_2, G_3] = 0.$$

The algebra is solvable.

Summing up the Lie series we obtain one-parameter subgroups of the symmetry group of transformations corresponding to the generators (7). For G_1 we have translations in time:

$$\widetilde{t} = t + \epsilon, \qquad \widetilde{r} = r, \qquad \widetilde{u} = u, \qquad \widetilde{f_\alpha} = f_\alpha, \qquad \widetilde{E} = E.$$

This symmetry follows from the fact that coefficients of equations (5) do not depend on variable t. The G_2 and G_3 generate the following scaling transformations:

$$\widetilde{t} = t\exp(-\epsilon), \qquad \widetilde{r} = r\exp(-2\epsilon), \qquad \widetilde{u} = u\exp(-\epsilon), \qquad \widetilde{f_\alpha} = f_\alpha \exp(5\epsilon), \qquad \widetilde{E} = E,$$

and

$$\widetilde{t} = t\exp(-3\epsilon), \qquad \widetilde{r} = r\exp(-\epsilon), \qquad \widetilde{u} = u\exp(2\epsilon), \qquad \widetilde{f_\alpha} = f_\alpha, \qquad \widetilde{E} = E\exp(5\epsilon).$$

The lack of Galilean symmetry and spatial translational symmetry, contrary to the case of one-dimensional rectilinear motion [6–8], follows from the fact that for radial motion there exists the distinguished frame of reference connected with the point $r = 0$.

3 Classification of invariant solutions

Using the adjoint representation we find the optimal system of one-dimensional subalgebras (see [9, 10]). They are generated by the following operators:

$$G_1, \quad G_2, \quad G_3, \quad \pm G_1 - 3G_2 + G_3, \quad a_2G_2 + a_3G_3 \ \text{ for arbitrary } a_2, a_3 \neq 0.$$

Since G_2 commutes with G_3, we can express the finite transformations corresponding to the generator $a_2G_2 + a_3G_3$ as compositions of one-dimensional subgroups generated by operators G_2

and G_3. Similarly, because $[\pm G_1, -3G_2 + G_3] = 0$ the finite transformation corresponding to $\pm G_1 - 3G_2 + G_3$ is a composition of subgroups generated by G_1, G_2 and G_3. Using these transformations we find invariants built from independent and dependent variables. From these invariants follow the forms of invariant solutions corresponding to the optimal system. For example,

$$e^{\epsilon(\pm G_1 - 3G_2 + G_3)} = e^{\epsilon \pm G_1} e^{\epsilon(G_3 - 3G_2)} = e^{\pm\epsilon G_1} e^{-3\epsilon G_2} e^{\epsilon G_3}$$

leads to the following transformation

$$\widetilde{t'} = t' e^{\pm\epsilon}, \qquad \widetilde{r} = r e^{5\epsilon}, \qquad \widetilde{u} = u e^{5\epsilon}, \qquad \widetilde{f_\alpha} = f_\alpha e^{-15\epsilon}, \qquad \widetilde{E} = E e^{5\epsilon},$$

where $t' = e^t$. Invariants have the forms

$$r e^{\mp 5t}, \quad u e^{\mp 5t}, \quad r u^{-1}, \quad f_\alpha e^{\pm 15t}, \quad r^3 f_\alpha, \quad u^3 f_\alpha, \quad E e^{\mp 5t}, \quad r^{-1} E, \quad u^{-1} E, \quad \ldots.$$

We choose independent invariants (two of them built from independent variables – see [11])

$$y = r e^{\mp 5t}, \qquad z = u e^{\mp 5t}, \qquad f_{\alpha 0} = f_\alpha e^{\pm 15t}, \qquad E_0 = E e^{\mp 5t}$$

and look for solutions in the form

$$f_\alpha = e^{\mp 15t} f_{\alpha 0}\big(r e^{\mp 5t}, u e^{\mp 5t}\big), \qquad E = e^{\pm 5t} E_0\big(r e^{\mp 5t}\big),$$

because electric field does not depend on velocity u.

The results of this classification of essentially independent invariant solutions are gathered in the following table

No	Subgroup	Form of the solution
1	G_1	$f_\alpha(r,u)\,,\ E(r)$
2	G_2	$t^{-5} f_\alpha\big(t^2 r^{-1}, r^{-1} u^2\big), \quad E\big(t^2 r^{-1}\big)$
3	G_3	$f_\alpha\big(t r^{-3}, r^{-1} u^2\big), \quad t^{-1} r^{-2} E\big(t r^{-3}\big)$
4	$\pm G_1 - 3G_2 + G_3$	$e^{\mp 15t} f_\alpha\big(r e^{\mp 5t}, u e^{\mp 5t}\big), \quad e^{\pm 5t} E\big(r e^{\mp 5t}\big)$
5	$a_2 G_2 + a_3 G_3$	$r^{-2} u f_\alpha\big(t^{(2a_2+a_3)} r^{-(a_2+3a_3)}, t r^{-1} u\big),$ $t^{-2} r E\big(t^{(2a_2+a_3)} r^{-(a_2+3a_3)}\big)$

For example, stationary solutions $f_\alpha(r,u)$, $E(r)$ corresponding to G_1 are generalizations of nonlinear electrostatic BGK-waves (Bernstein, Greene, Kruskal [15]). The BGK-solutions are found by solving an integro-differential equation for electrostatic potential. This equation, which is obtained [2] by substituting the general solution of the Vlasov equation into the Poisson equation, is now more complicated due to the term $2Er^{-1}$. In the case of vanishing field $E = 0$ we obtain the equilibrium solutions of the same form as in the rectilinear case: $f_\alpha(u)$, where f_α is an arbitrary function.

4 Conclusions

The results obtained in this paper provide a new example of effectiveness of the method presented in [6–8]. Namely, we have found symmetry groups for Vlasov–Maxwell equations for spherically symmetric systems of multi-component collisionless plasma. The optimal system of one-parameter subgroups is found and classification of corresponding invariant solutions is carried out.

Spherical symmetry of our problem leads to determining equations, which are different from the case of one-dimensional rectilinear motion due to higher powers of velocity in the Vlasov–Maxwell equations.

The symmetry group is narrower, only three-dimensional, than the symmetry group for one-dimensional rectilinear case [6,7,14]. The lack of the Galilean symmetry and spatial translational symmetry for the radial case is obvious since for radial motion there exists the distinguished frame of reference connected with the point $r = 0$. Other differences are of minor importance.

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