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On ∗-Representations of Algebras of Temperley–Lieb Type and Algebras Generated by Linearly Dependent Generators with Given Spectra

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Algebras of Temperley–Lieb type and algebras generated by linearly dependent generators with a given spectra are presented in this paper. We consider their ∗-representations and sets of parameters, for which ∗-representations exist. Examples of algebras are considered.

1 Introduction

Let $\{A_k\}_{k=1}^n$ be a set of linear operators in separable complex Hilbert space H with scalar sum $\sum\limits_{k=1}^{n} A_k = \lambda I_H$ and a restriction that spectrum of each A_k belongs to a certain finite set $M_k \subset \mathbb{C}$. Such sets of operators play an important role in analysis, algebraic geometry, operator theory and mathematical physics [1–3].

Algebras, generated by linearly connected generators with given spectra were introduced and studied in [5–10]. Particularly, results on their growth, existence of polynomial identities, representations etc. where obtained. Following scheme presented in [11, 12]. In Sections 2, 3 we introduce connections of these algebras with Temperley–Lieb algebras. We study their representations and sets of parameters for which representations exist. In Sections 4–6 we consider examples of algebras including algebras connected with extended Dynkin diagrams.

2 Homomorphisms of algebras $\mathcal{P}_{M_1,\ldots,M_n;\lambda}$ and $\mathcal{P}_{N_1,\ldots,N_n,abo}$

Consider ∗-algebra, generated by n self-adjoint elements a_i for $i = 1, \ldots, n$ which satisfy corresponding restrictions on spectra $\sigma(a_i) \subseteq \{x_0^{(i)}, \ldots, x_{m_i}^{(i)}\} = M_i$ for given finite sets of real numbers $x_0^{(i)} < \cdots < x_{m_i}^{(i)}$ and relation $\sum\limits_{i=1}^{n} a_i = \lambda e$, $\lambda \in \mathbb{R}$.

Without loss of generality we shall think, that $x_k^{(i)} > 0$ and $x_0^{(i)} = 0$ where $i = 1, \ldots, n$, $k = 1, \ldots, m_i$. We consider $\lambda > 0$ (for if $\lambda < 0$ representations of such algebra do not exist and if $\lambda = 0$ there are only trivial ones).

Each element a_i can be presented in a form $a_i = \sum\limits_{k=1}^{m_i} x_k^{(i)} p_k^{(i)}$, where $p_k^{(i)}$-projection, $p_k^{(i)} p_l^{(i)} = 0$, $k \neq l$, $k, l = 1, \ldots, m_i$.

In [5] the following algebra generated by projections was introduced:

$$\mathcal{P}_{M_1,\ldots,M_n;\lambda} = \mathbb{C}\langle p_1^{(1)}, \ldots, p_{m_1}^{(1)}, p_1^{(2)}, \ldots, p_{m_2}^{(2)}, \ldots, p_1^{(n)}, \ldots, p_{m_n}^{(n)} \mid \sum_{i=1}^{n} \sum_{k=1}^{m_i} x_k^{(i)} p_k^{(i)} = \lambda e,$$
$$p_k^{(i)} p_l^{(i)} = 0 \ (l \neq k),\ p_k^{(i)2} = p_k^{(i)},\ l, k = 1, \ldots, m_i,\ i = 1, \ldots, n\rangle.$$

With every such algebra we associate a graph (or diagram) $G = G(\mathcal{P}_{M_1,\ldots,M_n;\lambda})$ which consists of one root vertex and n branches, i-th branch is a sequence of Card $(M_i) - 1$ connected vertices.

We mark vertices of i-th branch with nonzero real numbers of set M_i starting from root in descending order. Also we mark root vertex with λ.

Define $N_j = \frac{1}{\lambda} M_j = \left\{ 0 < \frac{x_1^{(j)}}{\lambda} < \cdots < \frac{x_{m_j}^{(j)}}{\lambda} \right\}$, $j = 1, \ldots, n$. Consider algebra $\mathcal{P}_{N_1,\ldots,N_n,abo}$ generated by projections p and $q_k^{(i)}$, where $k = 1, \ldots, m_i$, $i = 1, \ldots, n$

$$\mathcal{P}_{N_1,\ldots,N_n,abo} = \mathbb{C}\langle q_1^{(1)}, \ldots, q_{m_1}^{(1)}, q_1^{(2)}, \ldots, q_{m_2}^{(2)}, \ldots, q_1^{(n)}, \ldots, q_{m_n}^{(n)}, p \mid \sum_{i=1}^{n} \sum_{k=1}^{m_i} q_k^{(i)} = e,$$
$$q_k^{(i)} p q_k^{(i)} = \frac{x_k^{(i)}}{\lambda} q_k^{(i)},\ q_k^{(i)} p q_l^{(i)} = 0\ (l \neq k),\ p^2 = p,\ q_k^{(i)2} = q_k^{(i)},$$
$$q_k^{(i)} q_s^{(j)} = 0\ ((i,k) \neq ((j,s)),\ k,l = 1, \ldots, m_i,\ s = 1, \ldots, m_j,\ i,j = 1, \ldots, n\rangle.$$

These algebras are called *abo*-analogs (“*all but one*”).

Proposition 1. *There exists a homomorphism of algebras*

$$\varphi_1 : \mathcal{P}_{M_1,\ldots,M_n;\lambda} \longrightarrow p\mathcal{P}_{N_1,\ldots,N_n,abo}p,$$

which is defined on generators in the following way

$$\varphi_1\big(p_k^{(i)}\big) = \frac{\lambda}{x_k^{(i)}} p q_k^{(i)} p.$$

Assume $\mathrm{q} = \mathrm{diag}\,\big(p_1^{(1)}, \ldots, p_{m_1}^{(1)}, \ldots, p_1^{(n)}, \ldots, p_{m_n}^{(n)}\big) \in \mathrm{M}_m(\mathcal{P}_{M_1,\ldots,M_n;\lambda})$, $m = \sum\limits_{i=1}^{n} m_i$ and $e_{i,j}$ is matrix unity in $\mathrm{M}_m(\mathcal{P}_{M_1,\ldots,M_n;\lambda})$.

Proposition 2. *There exists a homomorphism of algebras*

$$\varphi_2 : \mathcal{P}_{N_1,\ldots,N_n,abo} \longrightarrow q\mathrm{M}_m(\mathcal{P}_{M_1,\ldots,M_n;\lambda})q,$$

which is defined on generators in the following way

$$\varphi_2\big(q_k^{(i)}\big) = p_k^{(i)} \otimes \mathrm{e}_{k+t_i,k+t_i},$$
$$\varphi_2(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{m_i} \sum_{l=1}^{m_j} \frac{\sqrt{x_k^{(i)} x_l^{(j)}}}{\lambda} p_k^{(i)} p_l^{(j)} \otimes \mathrm{e}_{k+t_i,l+t_j},$$

where $t_i = \sum\limits_{j<i} m_j$.

Remark 1. If we introduce involution “$*$” on algebras $\mathcal{P}_{M_1,\ldots,M_n;\lambda}$ and $\mathcal{P}_{N_1,\ldots,N_n,abo}$ in the obvious way (by assuming all generators to be self-adjoint), then homomorphisms φ_1 and φ_2 appear to be $*$-homomorphisms of $*$-algebras.

3 Equivalence of categories Rep $\mathcal{P}_{M_1,\ldots,M_n;\lambda}$ and Rep $\mathcal{P}_{N_1,\ldots,N_n,abo}$

With the help of homomorphisms φ_1 and φ_2 it is possible to build corresponding functors on categories of representations [11].

Given a homomorphism of algebras $\varphi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$, there exists a functor $F : \mathrm{Rep}\,\mathcal{A}_2 \longrightarrow \mathrm{Rep}\,\mathcal{A}_1$ defined by rules $F(\pi) = \pi \circ \varphi$ and $F(K) = K$, where $\pi \in \mathrm{Ob}\,(\mathrm{Rep}\,\mathcal{A}_2)$ and K is a morphism in the category $\mathrm{Rep}\,\mathcal{A}_2$.

Assume $\mathcal{A}$ to be an algebra and q to be an idempotent in $\mathcal{A}$. Then $\mathcal{B} = q\mathcal{A}q$ is algebra with unit q. For $\pi \in \operatorname{Rep}\mathcal{A}$ we can define representation $\widehat{\pi} \in \operatorname{Rep}\mathcal{B}$ in space $\operatorname{Im}\pi(q)$ by $\widehat{\pi}(x) = \pi(x)\,|_{\operatorname{Im}\pi(q)}$ for all $x \in \mathcal{B}$. If K is the intertwining operator between representations π_1 and π_2, then $K\,|_{\operatorname{Im}\pi(q)}$ is the intertwining operator between representations $\widehat{\pi_1}$ and $\widehat{\pi_2}$. Thus we constructed a functor from the category $\operatorname{Rep}\mathcal{A}$ to $\operatorname{Rep}\mathcal{B}$.

Let $\mathcal{A}$ be an algebra, $\pi : \mathcal{A} \longrightarrow L(H)$ be a representation in space H. Construct a functor $F : \operatorname{Rep}\mathcal{A} \longrightarrow \operatorname{Rep} M_n(\mathcal{A})$ (where $M_n(\mathcal{A}) = \mathcal{A} \otimes M_n(\mathbb{C})$ is the algebra of matrices over $\mathcal{A}$). Define a representation $\widetilde{\pi} : M_n(\mathcal{A}) \longrightarrow L(H \oplus \cdots \oplus H)$ by $\widetilde{\pi}(a \otimes b) = \pi(a) \otimes b$, where $a \in \mathcal{A}$ and $b \in M_n(\mathbb{C})$. Put $F(\pi) = \widetilde{\pi}$ and $F(K) = K \otimes I_n$, where K is a morphism in the category $\mathcal{A}$ and I_n is the identity operator in $M_n(\mathcal{A})$.

Given two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, let q be an idempotent in $M_n(\mathcal{A}_2)$. For a homomorphism of algebras $\varphi : \mathcal{A}_1 \longrightarrow qM_n(\mathcal{A}_2)q$, with the previous techniques a functor $F_\varphi : \operatorname{Rep}\mathcal{A}_2 \longrightarrow \operatorname{Rep}\mathcal{A}_1$ can be easily constructed. If $\pi : \mathcal{A}_2 \longrightarrow L(H)$, then $F_\varphi(\pi) : \mathcal{A}_1 \longrightarrow L(\mathcal{H})$, where $\mathcal{H} = \widetilde{\pi}(q)(H \oplus \cdots \oplus H)$. We will identify the algebra $L(\mathcal{H})$ with an algebra of operators A in $L(H \oplus \cdots \oplus H)$, such that $\widetilde{\pi}(q)A = A\widetilde{\pi}(q) = A$. If K is a morphism in the category $\operatorname{Rep}\mathcal{A}_2$, then $F_\varphi(K) = (K \otimes I_n)|_{\mathcal{H}}$.

Proposition 3. *Functors F_{φ_1} and F_{φ_2} give equivalence of the categories* $\operatorname{Rep}\mathcal{P}_{M_1,\dots,M_n;\lambda}$ *and* $\operatorname{Rep}\mathcal{P}_{N_1,\dots,N_n,abo}$.

Remark 2. Proposition 3 remains true if we replace categories of representations with categories of $*$-representations of corresponding $*$-algebras with involution defined as in Remark 1.

Let

$$\mathrm{W} = \{\lambda \in \mathbb{R} \mid *-\operatorname{Rep}\mathcal{P}_{M_1,\dots,M_n;\lambda} \neq \varnothing\}, \qquad \widetilde{\mathrm{W}} = \left\{\lambda > 0 \mid *-\operatorname{Rep}\mathcal{P}_{\frac{M_1}{\lambda},\dots,\frac{M_n}{\lambda},abo} \neq \varnothing\right\}.$$

Theorem 1. $\mathrm{W} = \widetilde{\mathrm{W}} \cup \{0\}$.

4 Sums of projections

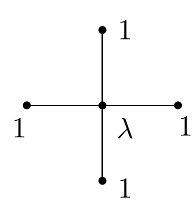
Let M_i for $i = 1,\dots,n$ be a sequence of sets of nonnegative real numbers containing zero. We denote by $W_{M_1,\dots,M_n}$ the set of all real numbers λ such that there exists a separable Hilbert space H and a sequence A_i, $i = 1,\dots,n$ of self-adjoint operators in H such that $\sum\limits_{i=1}^{n} A_i = \lambda I_H$ and $\sigma(A_i) \subset M_i$, i.e. there exist $*$-representations of $*$-algebra $\mathcal{P}_{M_1,\dots,M_n;\lambda}$. Let $\widetilde{W}_{M_1,\dots,M_n}$ be the corresponding set of all $\lambda > 0$ for which $*$-representations of $\mathcal{P}_{\frac{M_1}{\lambda},\dots,\frac{M_n}{\lambda},abo}$ exist together with a single element 0. Theorem 1 states that $\widetilde{W}_{M_1,\dots,M_n} = W_{M_1,\dots,M_n}$

Consider several examples of description of sets $W_{M_1,\dots,M_n}$ and $\widetilde{W}_{N_1,\dots,N_n}$ for some sequences (M_i), $i = 1,\dots,n$.

1. Let $M_1 = \cdots = M_4 = \{0,1\}$, then

$$\mathcal{P}_{M_1,M_2,M_3,M_4;\lambda} = \mathbb{C}\langle p_1, p_2, p_3, p_4 \mid p_1 + p_2 + p_3 + p_4 = \lambda e,\ p_i^2 = p_i^* = p_i\rangle,$$

and $*$-representations of such algebras are orthogonal projections with sum λI_H. The algebra $\mathcal{P}_{M_1,M_2,M_3,M_4;\lambda}$ corresponds to the diagram $\widetilde{D}_4$.



The *abo*-analog for this algebra is

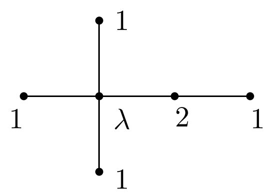
$$\mathcal{P}_{N_1,N_2,N_3,N_4,abo} = \mathbb{C}\langle q_1, q_2, q_3, q_4, p \mid q_1 + q_2 + q_3 + q_4 = e,\ q_i p q_i = \frac{1}{\lambda} q_i,$$
$$q_i^2 = q_i^* = q_i,\ i = 1, 2, 3, 4,\ p^2 = p^* = p,\ q_i q_j = 0\ (i \neq j)\rangle.$$

Description of set W_{M_1,M_2,M_3,M_4} was presented in [4]. We present it in our terms in the following proposition.

Proposition 4. *Assume* $M_1 = M_2 = M_3 = M_4 = \{0, 1\}$ *and* $S = \left\{\frac{1}{2},\ 1\right\}$. *Then*

$$\widetilde{W}_{M_1,M_2,M_3,M_4} = W_{M_1,M_2,M_3,M_4} = \left\{2 \pm \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in S\right\} \cup \{2\}.$$

2. Let $M_1 = M_2 = M_3 = \{0, 1\}$, $M_4 = \{0, 1, 2\}$. Algebra $\mathcal{P}_{M_1,M_2,M_3,M_4;\lambda}$ corresponds to the diagram



Description of W_{M_1,M_2,M_3,M_4} can be found in [9]. In our terms:

Proposition 5. *If* $M_1 = M_2 = M_3 = \{0, 1\}$, $M_4 = \{0, 1, 2\}$ *then*

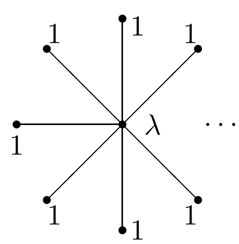
$$\widetilde{W}_{M_1,M_2,M_3,M_4} = W_{M_1,M_2,M_3,M_4} = \Lambda_4 \cup [2, 3] \cup (5 - \Lambda_4),$$

where $\Lambda_4 = \left\{2 - \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in \left\{\frac{1}{2}, 1\right\}\right\}$.

3. Consider $*$-algebra generated by n projections with scalar sum. Let $M_1 = \cdots = M_n = \{0, 1\}$, then

$$\mathcal{P}_{M_1,\ldots,M_n;\lambda} = \mathbb{C}\langle p_1, \ldots, p_n \mid \sum_{i=1}^{n} p_i = \lambda e,\ p_i^2 = p_i^* = p_i\rangle.$$

Its corresponding diagram is:



From [6] we conclude the following:

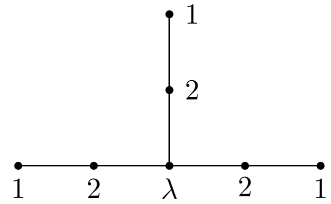
Proposition 6. *If* $M_1 = \cdots = M_n = \{0, 1\}$ *then*

$$\widetilde{W}_{M_1,\ldots,M_n} = W_{M_1,\ldots,M_n} = \left\{\Lambda_n^{(0)}, \Lambda_n^{(1)}, \left[\frac{n-\sqrt{n^2-4n}}{2}, \frac{n+\sqrt{n^2-4n}}{2}\right], n - \Lambda_n^{(0)}, n - \Lambda_n^{(1)}\right\},$$

where $\Lambda_n^{(i)} = \left\{i, 1 + \frac{1}{n-1-i}, 1 + \frac{1}{n-2-\frac{1}{n-1-i}}, 1 + \frac{1}{n-2-\frac{1}{n-2-\frac{1}{n-1-i}}}, \ldots\right\}$ *for* $i = 0, 1,\ n \in \mathbb{N}$.

5 Sums of operators with spectrum $\{0, 1, 2\}$

1. Let $M_1 = M_2 = M_3 = \{0, 1, 2\}$. Consider algebra $\mathcal{P}_{M_1,M_2,M_3;\lambda}$. Corresponding diagram is $\widetilde{E}_6$:

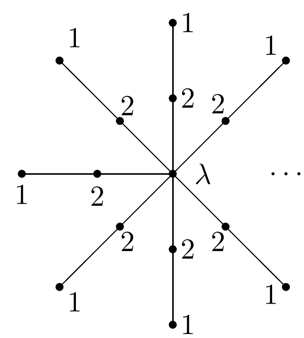


Description of W_{M_1,M_2,M_3} was given in [9]:

Proposition 7. *If* $M_1 = M_2 = M_3 = \{0, 1, 2\}$*, let* $S = \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$*. Then*

$$\widetilde{W}_{M_1,M_2,M_3} = W_{M_1,M_2,M_3} = \left\{3 \pm \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in S\right\} \cup \{3\}.$$

2. Consider $*$-algebra $\mathcal{P}_{M_1,\dots,M_n;\lambda}$ for $M_1 = \cdots = M_n = \{0, 1, 2\}$, it corresponds to the diagram:

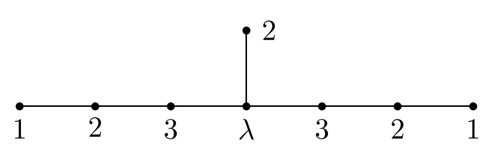


Proposition 8 ([9]). *If* $M_1 = \cdots = M_n = \{0, 1, 2\}$*, then*

$$\begin{aligned}\widetilde{W}_{M_1,\dots,M_n} &= W_{M_1,\dots,M_n} = ([0,2] \cap \Sigma_n) \cup [2, 2n-2] \cup (2n - ([0,2] \cap \Sigma_n)) \\ &= \Lambda_n^{(0)} \cup \Lambda_n^{(1)} \cup \left[n - \frac{n + \sqrt{n^2 - 4n}}{2}, n + \frac{n + \sqrt{n^2 - 4n}}{2}\right] \cup \big(2n - \Lambda_n^{(0)}\big) \cup \big(2n - \Lambda_n^{(1)}\big).\end{aligned}$$

6 Extended Dynkin diagrams E_7 and E_8

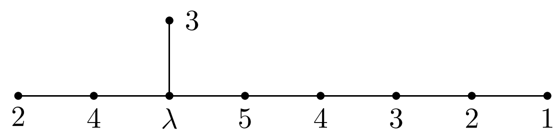
1. Let $M_1 = M_2 = \{0, 1, 2, 3\}$, $M_3 = \{0, 2\}$. Considering algebra $\mathcal{P}_{M_1,M_2,M_3;\lambda}$ we get diagram $\widetilde{E}_7$:



Proposition 9. *If* $M_1 = M_2 = \{0, 1, 2, 3\}$*,* $M_3 = \{0, 2\}$ *and* $S = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\}$ *then*

$$\widetilde{W}_{M_1,M_2,M_3} = W_{M_1,M_2,M_3} = \left\{4 \pm \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in S\right\} \cup \{4\}.$$

2. Let $M_1 = \{0,1,2,3,4,5\}$, $M_2 = \{0,2,4\}$ and $M_3 = \{0,3\}$. Consider $\mathcal{P}_{M_1,M_2,M_3;\lambda}$ Its corresponding diagram is $\widetilde{E}_8$



Proposition 10. *If* $M_1 = \{0,1,2,3,4,5\}$, $M_2 = \{0,2,4\}$, $M_3 = \{0,3\}$ *and* $S = \left\{\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, 1\right\}$ *then*

$$\widetilde{W}_{M_1,M_2,M_3} = W_{M_1,M_2,M_3} = \left\{6 \pm \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in S\right\} \cup \{6\}.$$

[1] Ewans D. and Kawahigashi Y., Quantum symmetries on operator algebras, New York, Oxford Univ. Press, 1998.

[2] Fulton W., Eigenvalues, invariant factors, highest weights, and Schubert calculus, *Bull. of the AMS*, 2000, V.37, N 3, 209–249.

[3] Klyachko A.A., Stable bundles, representation theory and Hermitian operators, *Selecta Math.*, 1998, V.4, 419–445.

[4] Ostrovsky V.L. and Samoilenko Yu.S., Introduction to the theory representation of finitely presented ∗-algebras. 1. Representations by bounded operators, *Rev. Math. & Math. Phys.*, 1999, V.11, 1–261.

[5] Rabanovich V.I. and Samoilenko Yu.S., When a sum of idempotents or projectors is a multiple of identity, *Funktsional. Anal. i Prilozhen.*, 2000, V.34, N 4, 91–93.

[6] Kruglyak S.A., Rabanovich V.I. and Samoilenko Yu.S., On sums of projectors, *Funktsional. Anal. i Prilozhen.*, 2002, V.36, N 3, 30–35.

[7] Kruglyak S.A. and Roiter A.V., Local-scalar reppresentations of graphs in category Hilbert spaces, Preprint, Kyiv, Institute of Mathematics of NAS of Ukraine, 2003.

[8] Mellit A.S., On the case where a sum of three partial reflections is equal to zero, *Ukr. Math. J.*, 2003, V.55, N 9, 1277–1283.

[9] Mellit A.S., Rabanovich V.I. and Samoilenko Yu.S., When sum partial reflection multiple to identity operator, *Funktsional. Anal. i Prilozhen.*, 2004, V.38, N 2, 80–82.

[10] Vlasenko M.A., Mellit A.S. and Samoilenko Yu.S., On algebras, generated by linearly connected generators with given spectrum, *Functsional. Anal. i Prilozhen.*, 2004, V.38, N 4, 33–42.

[11] Popovych S.V. and Samoilenko Yu.S., On homomorphisms of algebras generated by projections and Coxeter functors, *Ukr. Math. J.*, 2003, V.55, N 9, 1224–1237.

[12] Zavodovsky M.V., On algebras of Temperley–Lieb type, *Ukr. Math. J.*, 2004, V.59, N 5, 736–743.