Construction of Exact Solutions of Diffusion Equation

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New exact solutions for the heat equation $u_t = u_{xx} + \lambda u^{\frac{n-1}{2}} + \varepsilon u^{\frac{n+1}{2}} + cu + du^{\frac{3-n}{2}}$ are found.

1 Introduction

The nonlinear diffusion equations

$$u_t = u_{xx} + b(u)u_x + c(u), \tag{1}$$

where $u = u(t,x)$, $b(u)$, $c(u)$ are smooth functions, and subscripts denote derivatives with respect to the corresponding variables, play fundamental role in the modelling of various processes of heat conduction, reaction-diffusions, in mathematical biology, and also in many other spheres. Equation (1) generalizes a great number of known nonlinear evolution equations. Thus in particular cases equation (1) is regarded as classical Burgers equation

$$u_t = u_{xx} + \lambda_1 u u_x, \tag{2}$$

and also as Kolmogorov–Piskunov equation

$$u_t = u_{xx} + f(u), \tag{3}$$

where $f(u)$ is a sufficiently smooth function.

It is well known, that nonlocal Cole–Hopf transformation

$$u = -2\mu \frac{v_x}{v}, \qquad v = v(t,x) \tag{4}$$

makes every solution of linear heat conduction equation

$$v_t - \mu v_{xx} = 0 \tag{5}$$

correspond to the particular solution of the equation (2). By these means, the transformation (4) reduces the problem of construction of solutions of the equation (2) to the construction of solutions of linear equation (5).

The conditional symmetries equation (3) was investigated in [1, 2]. The operators of conditional symmetry of this equation were found only in the case when $f(u)$ is a polynomial of 3rd order. All known solutions of the equation (3) that were successfully obtained by means of conditional symmetry operators, have such representation

$$u = k\frac{z_x}{z}, \tag{6}$$

where k is constant, and $z = z(t,x)$ is some arbitrary function.

It should be noticed that group classification of equations (1) was done in [3]. The results concerning conditional symmetries of equation (1) are mentioned in [4]. In [5] wide classes of exact solutions of equation (3) are represented.

A goal of this article is to construct the exact solutions of equation

$$u_t = u_{xx} + \lambda u^{\frac{n-1}{2}} u_x + \varepsilon u^{\frac{n+1}{2}} + cu + du^{\frac{3-n}{2}}, \tag{7}$$

where $\varepsilon = \pm 1$, c, d are arbitrary real numbers. In case $n = 3$ we use transformation (3). The transformation (6) is particular case of transformation

$$u = k\left(\frac{z_x}{z}\right)^{\frac{2}{n-1}}, \tag{8}$$

that is effective in case of arbitrary n. By using the transformation (8) we reduce the problem of constructing of exact solutions of equation (7) to the problem of finding function $z = z(t,x)$. As will be mentioned further, function $z = z(t,x)$ is a solution of system of ordinary differential equations that is easily solved in many cases.

2 The exact solutions of equation (7) for $n = 3$

We construct the exact solutions of equation (7) for $n = 3$

$$u_t = u_{xx} + \lambda u u_x + \varepsilon u^2 + cu + d. \tag{9}$$

By substituting (6) into (9), we obtain

$$(kz_{xt} - kz_{xxx} - ckz_x - dz)z^2 + kz_x(-z_t + (3-\lambda k)z_{xx} - \varepsilon k z_x)z + (-2k + \lambda k^2) = 0.$$

The variable z will be determined from the conditions of zero expressions at z and z^2 simultaneously. As a result we have

$$kz_{xt} - kz_{xxxx} - ckz_x - dz = 0, \tag{10}$$
$$-z_t + (3-\lambda k)z_{xx} - \varepsilon k z_x = 0, \tag{11}$$
$$2 - \lambda k = 0. \tag{12}$$

From equation (11) and (12) we find

$$z_t = z_{xx} - \varepsilon k z_x. \tag{13}$$

By substituting into equation (10), we obtain

$$\varepsilon k^2 z_{xx} + ckz_x + dz = 0. \tag{14}$$

Thus, the system (10)–(12) is equivalent to the system (12)–(14). This system can be easily solved. The type of solution depends on roots of characteristic equation

$$\varepsilon k^2 r^2 + ckr + d = 0, \tag{15}$$

that corresponds to linear equation (14). The roots of characteristic equation (15) are $\frac{1}{k}m_1$, $\frac{1}{k}m_2$, where m_1 and m_2 are the roots quadratic equation

$$\varepsilon r^2 + cr + d = 0. \tag{16}$$

We consider tree cases.

a) The roots are real and different. The general solution of equation (14) has such form

$$z = \mu_1(t)\exp\left(\frac{1}{k}m_1 x\right) + \mu_2(t)\exp\left(\frac{1}{k}m_1 x\right),$$

where $\mu_1(t)$, $\mu_2(t)$ are functions of t which have to be found. By using equation (13), we obtain

$$\frac{d\mu_1}{dt} = \frac{1}{k^2}m_1^2\mu_1 - \varepsilon m_1\mu_1, \qquad \frac{d\mu_2}{dt} = \frac{1}{k^2}m_2^2\mu_2 - \varepsilon m_2\mu_2.$$

So,

$$\mu_1 = k_1\exp\left[\left(\frac{1}{k^2}m_1^2 - \varepsilon m_1\right)t\right], \qquad \mu_2 = k_2\exp\left[\left(\frac{1}{k^2}m_2^2 - \varepsilon m_2\right)t\right],$$

and

$$z = k_1\exp\left[\frac{1}{k}m_1 x + \left(\frac{1}{k^2}m_1^2 - \varepsilon m_1\right)t\right] + k_2\exp\left[\frac{1}{k}m_2 x + \left(\frac{1}{k^2}m_2^2 - \varepsilon m_2\right)t\right].$$

Thus we have found such solution of equation (2)

$$u = \frac{k_1 m_1\exp\left[\frac{\lambda}{2}m_1 x + \left(\frac{\lambda^2}{4}m_1^2 - \varepsilon m_1\right)t\right] + k_2 m_2\exp\left[\frac{\lambda}{2}m_2 x + \left(\frac{\lambda^2}{4}m_2^2 - \varepsilon m_2\right)t\right]}{k_1\exp\left[\frac{\lambda}{2}m_1 x + \left(\frac{\lambda^2}{4}m_1^2 - \varepsilon m_1\right)t\right] + k_2\exp\left[\frac{\lambda}{2}m_2 x + \left(\frac{\lambda^2}{4}m_2^2 - \varepsilon m_2\right)t\right]},$$

where k_1, k_2 are arbitrary real constants, that are not equal to zero simultaneously,

$$m_1 = \frac{-c_1 + \sqrt{c^2 - 4\varepsilon d}}{2\varepsilon}, \qquad m_2 = \frac{-c_1 - \sqrt{c^2 - 4\varepsilon d}}{2\varepsilon}.$$

b) The roots are complex numbers. Let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$. In this case we obtain

$$z = \exp\left[\frac{\lambda}{2}dx + \left(\frac{\lambda^2}{4}(\alpha^2 - \beta^2) - \varepsilon\alpha\right)t\right]$$
$$\times\left\{k_1\cos\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right] + k_2\sin\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right]\right\},$$

where k_1, k_2 are arbitrary real numbers, that are not equal to zero simultaneously. Thus, the solution of equation (9) has such form

$$u = \frac{\alpha k_1 + \beta k_2)\cos\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right] + (\alpha k_2 - \beta k_1)\sin\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right]}{k_1\cos\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right] + k_2\sin\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right]},$$

where

$$\alpha + i\beta = \frac{-c + \sqrt{c^2 - 4\varepsilon d}}{2\varepsilon}, \qquad \alpha - i\beta = \frac{-c - \sqrt{c^2 - 4\varepsilon d}}{2\varepsilon}.$$

c) The roots are equal. In this case $c^2 - 4\varepsilon d = 0$, and thus $d = \frac{c^2}{4\varepsilon}$, $m_1 = m_2 = -\frac{c}{2\varepsilon}$. Function z has such form

$$z = \exp\left[-\frac{\varepsilon\lambda c}{4}x + \left(\frac{\lambda^2 c^2}{16} + \frac{c}{2}\right)t\right]\left\{k_1 x + \left(-\frac{\varepsilon\lambda c}{2}k_1 - \frac{2\varepsilon}{\lambda}k_1\right)t + k_0\right\},$$

where k_1, k_2 are arbitrary real numbers, that are not equal to zero simultaneously. We obtain such solution of equation (9)

$$u = \frac{2}{\lambda}\frac{-\frac{\varepsilon\lambda c}{4}\left\{k_1 x + \left(-\frac{\varepsilon\lambda c}{2}k_1 - \frac{2\varepsilon}{\lambda}k_1\right)t + k_0\right\} + k_1}{k_1 x + \left(-\frac{\varepsilon\lambda c}{2}k_1 - \frac{2\varepsilon}{\lambda}k_1\right)t + k_0}.$$

In particular, if $c = 0$ then the solution has such form

$$u = \frac{2k_1}{\lambda k_1 x - 2\varepsilon k_1 t + \lambda k_0}.$$

Let us consider two separate cases.

a) $c = 0$, $\varepsilon < 0$. The equation (16) has two real and different roots

$$m_1 = \sqrt{-\varepsilon d}, \qquad m_2 = -\sqrt{-\varepsilon d}.$$

Thus the solution of equation (9) has form

$$u = \frac{k_1 m_1 \exp\left[\frac{\lambda m_1}{2}x + \left(-\frac{\varepsilon\lambda^2 d}{4} + \varepsilon m_2\right)t\right] + k_2 m_2 \exp\left[\frac{\lambda m_1}{2}x + \left(-\frac{\varepsilon\lambda^2 d}{4} + \varepsilon m_1\right)t\right]}{k_1 \exp\left[\frac{\lambda m_1}{2}x + \left(-\frac{\varepsilon\lambda^2 d}{4} + \varepsilon m_2\right)t\right] + k_2 \exp\left[\frac{\lambda m_2}{2}x + \left(-\frac{\varepsilon\lambda^2 d}{4} + \varepsilon m_1\right)t\right]}.$$

b) $c = 0$, $\varepsilon d > 0$. The equation (16) has two complex roots

$$m_1 = i\sqrt{\varepsilon d}, \qquad m_2 = -i\sqrt{\varepsilon d}.$$

Thus the solution of equation (9) has form

$$u = \frac{k_2\sqrt{\varepsilon d}\cos\left[\frac{\lambda}{2}\sqrt{\varepsilon d}x - \varepsilon\sqrt{\varepsilon d}t\right] - k_1\sqrt{\varepsilon d}\sin\left[\frac{\lambda}{2}\sqrt{\varepsilon d}x - \varepsilon\sqrt{\varepsilon d}t\right]}{k_1\cos\left[\frac{\lambda}{2}\sqrt{\varepsilon d}x - \varepsilon\sqrt{\varepsilon d}t\right] + k_2\sqrt{\varepsilon d}\sin\left[\frac{\lambda}{2}\sqrt{\varepsilon d}x - \varepsilon\sqrt{\varepsilon d}t\right]}.$$

3 The exact solutions of equation (7) for arbitrary n

Let us consider equation (7) for arbitrary n and for $d = e$

$$u_t = u_{xx} + \lambda u^{\frac{n-1}{2}}u_x + \varepsilon u^{\frac{n+1}{2}} + cu. \tag{17}$$

Substituting (8) into equation (17), we obtain such system for finding variable z

$$\frac{2}{n-1}z_x z_{xt} - \frac{2(3-n)}{(n-1)^2}z_{xx}^2 - \frac{2}{n-1}z_x z_{xx} - cz_x^2 = 0, \tag{18}$$

$$z_t = \left(\frac{n+3}{n-1} - \lambda k^{\frac{n-1}{2}}\right)z_{xx} - \varepsilon\frac{n-1}{2}k^{\frac{n-1}{2}}z_x, \tag{19}$$

$$k^{\frac{n-1}{2}} = \frac{n+1}{\lambda(n-1)}. \tag{20}$$

Solving system (18)–(20), we find such solution for equation (17)

$$u = \left[\frac{n+1}{\lambda(n-1)}\right]^{\frac{2}{n-1}} \frac{\exp\left[-\frac{2\varepsilon\lambda c}{n+1}x + \left(\frac{4\lambda^2c^2}{(n+1)^2} + c\right)t\right]}{\left\{-\frac{\varepsilon(n+1)}{\lambda(n-1)c}\exp\left[-\frac{\varepsilon\lambda c(n-1)}{n+1}x + \left(\frac{2\lambda^2c^2}{(n+1)^2} + \frac{c}{2}\right)(n-1)t\right] + c_1\right\}^{\frac{2}{n-1}}},$$

where c, c_2 are arbitrary constants.

This approach can be used for finding the exact solutions of more general type of reaction-diffusion equation that will be done in the next articles.

[1] Fushchych W.I. and Serov N.I., Conditional invariance and reduction of a nonlinear heat equation, *Proc. Acad. of Sci. Ukraine, Ser. A*, 1990, N 7, 24–27 (in Russian).

[2] Clarkson P.A. and Mansfield E.L., Symmetry reductions and exact solutions of a class of nonlinear heat equations, *Physica D*, 1994, V.70, N 3, 250–288.

[3] Dorodnitsyn V.A., On invariant solutions of non-linear heat equation with a sourse, *Zhurn. Vych. Matemat. Matemat. Fiziki*, 1982, V.22, N 6, 1393–1400 (in Russian).

[4] Cherniha N.D., Conditional symmetry of the Burgers equation and some of its generalizations, in Symmetry and Analytic Methods in Mathematical Physics, Kiyv, Institute of Mathematics, 1998, 265–269 (in Ukrainian).

[5] Nikitin A.G. and Barannyk T.A., Solitary wave and other solutions for nonlinear heat equations, math-ph/0303004.