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Finite-Dimensional Reductions of 2D dToda Hierarchy Constrained by the String Equation

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The paper concerns the study of rational and logarithmic reductions of the 2D dispersionless Toda hierarchy of integrable equations. The subject is motivated by important applications to problems in interface dynamics and statistical physics. We prove the consistency of such reductions with respect to the "Toda–Krichever" flows of the 2dToda hierarchy constrained by a string equation.

1 Introduction

Our paper concerns with the study of rational solutions of the dToda hierarchy of integrable equations. The subject is motivated by numerous important applications to problems of interface dynamics and statistical physics. We outline them briefly in this section and widen the discussion in the summary of [1] of the present volume.

The Laplacian growth is a process of propagation of a boundary between two phases on the plane [6,4]. The both phases are described by scalar harmonic fields (e.g. electrostatic potential, density or pressure), while the speed of the interface is proportional to the gradient of the difference between the fields. In the case where the boundary is an analytic curve it is usual to introduce a time-dependent conformal mapping from the exterior of the unit circle in the "mathematical" (w) plane $z = z(w, x)$, $w = \exp(\sqrt{-1}\phi)$, $0 < \phi < 2\pi$ to the boundary curve $z = X + \sqrt{-1}Y$ on the "physical" plane. It is shown in [6, 7, 4] that the Laplacian growth problem is equivalent to solution of the following equation

$$\operatorname{Im}\left(\frac{\partial z}{\partial \phi}\frac{\partial \bar z}{\partial x}\right) = w\left(\frac{\partial z(w,x)}{\partial w}\frac{\partial \bar z(1/w,x)}{\partial x} - \frac{\partial z(w,x)}{\partial x}\frac{\partial \bar z(1/w,x)}{\partial w}\right) = 1, \tag{1}$$

where bar stands for the complex conjugation (and $\bar w = w^{-1}$ at the curve). Equation (1) is called the Laplacian Growth equation [8, 7].

It turns out that (1) plays an essential role in the theory of infinite-dimensional integrable hierarchies. In particular, the equivalence of the contour dynamics to the dispersionless limit of the integrable Toda hierarchy constrained by (1) was established in [4]. Equation (1) may be interpreted as a constraint on an infinite commuting set of dynamical systems defined in the space of one-parameter families of conformal maps. This constraint characterizes the fixed points of an "additional symmetry" [5] in which x is interpreted as the flow parameter.

In what follows, we consider finite-dimensional solutions of (1) in a context of 2dToda hierarchy of integrable equations. We study formal algebraic solutions of the problem, forgetting about complex structure and treating z, $\bar z$ as independent functions, and w as a formal variable.

2 Reductions of 2dToda hierarchy constrained by string equation

2.1 2dToda hierarchy and string equation

The dToda hierarchy is defined in terms of two functions $z(w,x)$ and $\bar{z}(w^{-1},x)$ of the form:

$$z(w,x) = r(x)w + \sum_{k=0}^{\infty} u_k(x)w^{-k}, \qquad \bar{z}\left(w^{-1},x\right) = r(x)w^{-1} + \sum_{k=0}^{\infty} \bar{u}_k(x)w^{k}, \tag{2}$$

where the coefficients $r(x)$, $u_k(x)$, $\bar{u}_k(x)$ are viewed as coordinate functionals on the phase space consisting of such pairs $z(w,x)$, $\bar{z}(w^{-1},x)$. The dToda flow equations are

$$\partial_{t_k} z = \{H_k, z\}, \qquad \partial_{\bar{t}_k} \bar{z} = \{\bar{H}_k, \bar{z}\}, \qquad \partial_{t_k} \bar{z} = \{H_k, \bar{z}\}, \qquad \partial_{\bar{t}_k} z = \{\bar{H}_k, z\}, \tag{3}$$

$$H_k = (z^k)_+ + 1/2\big(z^k\big)_0, \qquad \bar{H}_k = \big(\bar{z}^k\big)_- + 1/2\big(\bar{z}^k\big)_0 \tag{4}$$

with subscripts $\pm$, 0 denoting the negative/positive and zero parts of the formal Laurent expansion in w. The Poisson–Lax bracket notation here stays for

$$\{f,g\} := w\frac{\partial f}{\partial w}\frac{\partial g}{\partial x} - w\frac{\partial f}{\partial x}\frac{\partial g}{\partial w}. \tag{5}$$

The dispersionless limit of the "string equation" is the constraint

$$\left\{z(w,x), \bar{z}\left(w^{-1},x\right)\right\} = 1. \tag{6}$$

Equation (6) is invariant under (3) and so defines an invariant under the dToda flows (3) manifold and a reduction of the full dToda hierarchy. The reduction of the dToda hierarchy by the string equation is still an infinite, compatible set of infinite-dimensional dynamical systems. In what follows we are interested in further "functional" reductions where z, $\bar{z}$ are polynomial, rational or logarithmic functions of w. As we show below such reductions are consistent with 2D dToda flows (3) if the string equation (6) holds. Thus, for the consistency we need a double ("functional" plus "string") reduction, which defines a finite-dimensional invariant sub-manifold on the phase space of general 2d Toda hierarchy.

2.2 Polynomial reductions

We begin with polynomial reductions of the 2D Toda chain

$$z(w) = rw + \sum_{i=-N}^{0} u_i w^i, \qquad \bar{z}\left(w^{-1}\right) = rw^{-1} + \sum_{i=0}^{N} \bar{u}_i w^i \tag{7}$$

constrained by the string equation (6). The following proposition states consistency of the polynomial reductions under the Toda flows:

Proposition 1. *If the string equation* (6) *holds, then* (7) *belong to a manifold invariant under the 2dToda flows* $\partial/\partial t_i$, $\partial/\partial \bar{t}_i$, $0 < i < N+2$ (3)*. This manifold has dimension* $2N+3$ *with* $r = r(T)$, $u = u(T)$, $\bar{u} = \bar{u}(T)$, $T = \{t_i, \bar{t}_i\}$, $i = 1, \ldots, 2N+1$ *being solution of the dynamical system induced by 2dToda flows* (3)*.*

Proof. We must prove that z remains of the form (7) under the flows generated by H_k, $\bar{H}_k$ provided the string equation (6) holds. (The proof for $\bar{z}$ is similar).

1. First we proceed with the flows generated by H_k. The lowest degree of z is $-N$. Since

$$H_k = \left(z^k\right)_+ + 1/2\left(z^k\right)_0 = h_k w^k + h_{k-1}w^{k-1} + \cdots + h_0 \tag{8}$$

is a Laurent polynomial of positive degree, the lowest degree term in the bracket $\{z, H_k\}$ is not less than $-N$. Therefore the lowest degree term in z remains $\geq -N$ under the flows. On the other hand, the complement $z^k - H_k$ of (8) is a polynomial in $1/w$ and so, $\{H_k, z\} = -\{z^k - H_k, z\}$ is a Laurent polynomial with the highest degree in w not exceeding 1. Therefore the highest degree term in z remains ≤ 1.

2. Unlike the evolution under the H_k-flows, the form-invariance (7) of z under the flows generated by the $\bar{H}_k$'s requires extra restrictions on the derivatives $\partial_{t_k} u_i$, stemming from the string equation. Since $\bar{H}_k = (\bar{z}^k)_- + 1/2(\bar{z}^k)_0$ is a Laurent polynomial of non-positive degree, we again see that the highest degree in $\{z, H_k\}$ is 1, and hence this degree cannot increase under the flow. However, since $\bar{H}_k = \bar{z}^k - ((\bar{z}^k)_+ + 1/2(\bar{z}^k)_0)$ is the difference between $\bar{z}^k$ and a polynomial of nonnegative degree, $\{\bar{H}_k, z\} = \{\bar{z}^k, z\} - \{((\bar{z}^k)_+ + 1/2(\bar{z}^k)_0), z\}$, the second bracket in this expression has lowest degree $\geq -N$, but not, in general, the first. However, since $\{\bar{z}^k, z\} = k\bar{z}^{k-1}\{\bar{z}, z\}$, imposing the extra restriction (6) (string equation), again implies that the lowest degree term in this bracket does not exceed $-N$, provided $k \leq N+1$. ■

2.3 Rational reductions

Consider now the space of rational functions $z(w)$ and $\bar{z}(w)$ of the form

$$z(w) = \frac{q_{N+1}(w)}{p_N(w)} = \frac{rw^{N+1} + \sum\limits_{i=0}^{N} a_i w^i}{w^N + \sum\limits_{i=0}^{N-1} b_i w^i}, \tag{9}$$

$$\bar{z}(w^{-1}) = \frac{\bar{q}_{N+1}(w^{-1})}{\bar{p}_N(w^{-1})} = \frac{rw^{-(N+1)} + \sum\limits_{i=0}^{N} \bar{a}_i w^{-i}}{w^{-N} + \sum\limits_{i=0}^{N-1} \bar{b}_i w^{-i}}, \tag{10}$$

where the $4N+3$ coefficients r, a_i, $\bar{a}_i$, b_i, $\bar{b}_i$ are functions of x. The following Lemma states onvariance of such rational reductions of $z(w)$ and $\bar{z}(w)$ under the dToda flows.

Lemma 1. *The space of functions $z(w)$ of the form* (9) *is invariant under the dToda flows ∂_{t_i}, $i > 0$. and, similarly, the space of functions $\bar{z}(w^{-1})$ of the form* (10) *is invariant under the $\partial_{\bar{t}_i}$, $i > 0$ flows.*

Proof. Consider the flows generated by H_k. The proof for $\bar{z}$ is similar. Since

$$H_k = \left(z^k\right)_+ + 1/2\left(z^k\right)_0 \tag{11}$$

is a polynomial of nonnegative degree in w, its complement $z^k - H_k$ is a polynomial in $1/w$. Then the Laurent expansion of brackets (5) $\{H_k, z\} = -\{z^k - H_k, z\}$ around infinity has the following form

$$\{H_k, z\} = k_1(T)w + k_0(T) + k_{-1}(T)w^{-1} + \cdots . \tag{12}$$

While the Lax bracket (5) implies that it is rational of the form $\{H_k, z\} = Q(w)/p_N(w)^2$, it follows from (12) that the highest degree in polynomial $Q(w)$ does not exceed $2N+1$. On the other hand, since it is clear that $\partial_{t_k} z = P(w)/{p_N}^2(w)$, $P(w) = p_N \partial_{t_k} q_{N+1} - q_{N+1} \partial_{t_k} p_N$, where the highest degree in the polynomial $P(w)$ also does not exceed $2N+1$, we may therefore equate to zero the coefficients of polynomial $P(w)-Q(w)$, and thereby get a system of differential equations for r, a, b. The number of equations obtained is $2N+2$. Thus we get a compatible system of differential equations for $2N+2$ unknowns r, a, b. ■

The consistency of rational solutions for the whole Toda hierarchy requires some extra restrictions. This is the point where the string equation plays an essential role.

Proposition 2. *The string equation* (6) *is a sufficient condition for the rational functions* $z(w)$ *and* $\bar{z}(w)$ (9), (10) *to belong to a manifold, which is only invariant under the first two-Toda flows* $\partial/\partial t_1$, $\partial/\partial\bar{t}_1$ *(see* (3)*).*

Proof is by argument similar to those used for Lemma (1) and Proposition (1).

So, there are only two flows compatible with the rational ansatz. In fact, we could not expect more invariant flows associated to the two simple poles at $w=0$ and $w=\infty$. In the polynomial case, the number of invariant flows was equal to the number of variables (polynomial coefficients), since one can associate n invariant flows to the pole of the n th order, and poles at zero and infinity are immovable. Below, we introduce additional flows, related to movable singularities of $z(w)$ and $\bar{z}(w^{-1})$ using a result by Krichever.

2.4 Additional flows for rational reductions of the dKP hierarchy

As mentioned in [2], on the phase space of extended Benney systems, i.e. rational dKP reductions admitting poles of arbitrary degree, there arise some new flows related to the pole structure of the corresponding maps. These additional flows were introduced by Krichever (see [3]).

Consider a representation of rational maps in pole-residue form for simple poles along with a half (1dToda or dKP) of flows. In the Takasaki gauge [9, 2]

$$z(w) = w + u_0 + \sum_{\alpha=1}^{N} \frac{u_\alpha}{w - w_\alpha}. \tag{13}$$

The new flows attached to the poles are defined as before

$$\partial_{t_{k,\alpha}} z = \{B_{k,\alpha}, z\}, \qquad \alpha = \infty, 1, 2, \ldots, \quad k = 0, 1, 2, \ldots \tag{14}$$

with the evolution operators being associated with the pole structure of z as follows :

$$B_{k,\infty} = \left(z(w)^k\right)_{\geq 0}$$

for an immovable pole at infinity, while for each finite-distance pole there appear additional flows with evolution operators as follows:

$$B_{k,\alpha} = \left(z(w)^k\right)_\alpha, \qquad B_{0,\alpha} = \log(w_\alpha - w).$$

Here, $z(w)_\alpha$ stays for the negative part of a formal expansion of $z(w)$ near its poles w_α:

$$f(w)_\alpha = \sum_{i>0} \frac{f_i}{(w - w_\alpha)^i} \qquad \text{if} \qquad f = \sum_{i\in Z} \frac{f_i}{(w - w_\alpha)^i}. \tag{15}$$

These additional flows commute amongst themselves and with the ordinary (associated with poles at infinity) 1dToda or dKP flows [3].

2.5 Additional invariant flows of the 2dToda system

The dKP hierarchy can be extended to the 2dToda system by introducing an infinite set of $\bar{t}$-flows associated to the poles at $w=0$ and $w=\infty$, similarly the Krichever–Benney system has the following 2dToda extension:

$$z(w) = rw + u_0 + \sum_{\alpha=1}^{N} \frac{u_\alpha}{w - w_\alpha}, \qquad \bar{z}(1/w) = r/w + \bar{u}_0 + \sum_{\beta=1}^{N} \frac{\bar{u}_\beta}{1/w - \bar{w}_\beta}, \tag{16}$$

$$\partial_{t_{k,\alpha}} z = \{H_{k,\alpha}, z\}, \quad \partial_{\bar{t}_{k,\beta}} \bar{z} = \{\bar{H}_{k,\beta}, \bar{z}\}, \quad \partial_{t_{k,\alpha}} \bar{z} = \{H_{k,\alpha}, \bar{z}\}, \quad \partial_{\bar{t}_{k,\beta}} z = \{\bar{H}_{k,\beta}, z\}. \tag{17}$$

In (17) we transformed the flows from the Takasaki to the Lax–Sato gauge needed for our purposes. The relation between the evolution operators in different gauges is found to be:

$$H_{k,\alpha}(w) = B_{k,\alpha}(w) - \tfrac{1}{2} B_{k,\alpha}(w=0), \qquad \bar{H}_{k,\beta}(y) = \bar{B}_{k,\beta}(y) - \tfrac{1}{2}\bar{B}_{k,\beta}(y=0), \tag{18}$$

where

$$\begin{aligned} &B_{k,\infty}(w) = (z(w)^k)_{\geq 0}, \quad B_{0,\alpha} = \log\left(r(w_\alpha - w)\right), \quad B_{k,\alpha}(w) = (z(w)^k)_\alpha, \\ &\bar{B}_{k,\infty}(y) = (\bar{z}(y)^k)_{\geq 0}, \quad \bar{B}_{0,\beta} = \log\left(r(\bar{w}_\beta - y)\right), \quad \bar{B}_{k,\beta}(y) = (\bar{z}(y)^k)_\beta, \quad y = 1/w. \end{aligned} \tag{19}$$

Lemma 2. *All vector fields attached to the pole structure of rational maps formally commute, i.e. the introduced evolution operators* (18) *satisfy zero-curvature conditions.*

It is important to note that in the system (17), the equations for the flows $\partial_{t_k}\bar{z}$, $\partial_{\bar{t}_k} z$ do not make a sense fully, since these flows do not preserve the rational ansatz for z, $\bar{z}$ in (13) and (16) until the additional reduction is made. As we have seen before, to be consistent, these systems must be restricted by a string equation, which makes (17) a finite-dimensional dynamical system.

Proposition 3. *The* $4N+2$ *commuting Toda–Krichever flows*

$$\begin{aligned} &\partial_{\tau_i} z = \{h_i, z\}, \qquad \partial_{\bar{\tau}_i} z = \left\{\bar{h}_i, z\right\}, \\ &\partial_{\tau_i} \bar{z} = \{h_i, \bar{z}\}, \qquad \partial_{\bar{\tau}_i} \bar{z} = \left\{\bar{h}_i, \bar{z}\right\}, \qquad i = 0, \dots, 2N, \end{aligned} \tag{20}$$

where

$$\begin{aligned} &h_0 = H_{1,\infty} = rw + u_0/2, && \bar{h}_0 = \bar{H}_{1,\infty} = r/w + \bar{u}_0/2, \\ &h_{2i-1} = H_{1,i} = \frac{u_i}{w - w_i} + \frac{u_i}{2w_i}, && \bar{h}_{2i-1} = \bar{H}_{1,i} = \frac{\bar{u}_i}{\bar{w}_i - 1/w} + \frac{1}{2}\bar{u}_i/\bar{w}_i, \\ &h_{2i} = H_{0,i} = \log(w_i - w) + 1/2\log(r/w_i), && \bar{h}_{2i} = \bar{H}_{0,i} = \log(\bar{w}_i - 1/w) + 1/2\log(r/\bar{w}_i) \end{aligned} \tag{21}$$

and

$$\tau_0 = t_{1,\infty}, \quad \tau_{2i-1} = t_{1,i}, \quad \tau_{2i} = t_{0,i}, \quad \bar{\tau}_0 = \bar{t}_{1,\infty}, \quad \bar{\tau}_{2i-1} = \bar{t}_{1,i}, \quad \bar{\tau}_{2i} = \bar{t}_{0,i} \tag{22}$$

preserve the rational form of $z(w)$ (13) *and* $\bar{z}(1/w)$ (16) *(or equally* (10), (9)*) provided the string equation* (6) *holds.*

As in the polynomial case, the total number of invariant flows equals the dimension of the dynamical system minus one. In what follows we show that these flows are Hamiltonian. Since the dimension of the phase space is odd and is equal to $4N+3$, it is, in fact a Poisson manifold where the dimension of the symplectic leaf is $4N+2$. This last number is exactly equal to the number of commuting Toda–Krichever flows.

Now we conjecture that the stated result is true for the more general setting. We introduce a logarithmic ansatz for the 2dToda hierarchy and prove Lemma 2 and Proposition 3 for logarithms followed by a limiting procedure.

2.6 Logarithmic flows

As was mentioned above, it is easier to prove consistency of the rational ansatz with the dynamics of the 2dToda system using the more general Logarithmic solutions. Let us set:

$$\begin{aligned} z &= r(x)w + u(x) + \sum_{i=1}^{n+1} a_i \log(w_i(x) - w), \qquad \sum_{i=1}^{n+1} a_i = 0, \\ \bar{z} &= r(x)w^{-1} + \bar{u}(x) + \sum_{i=1}^{n+1} \bar{a}_i \log\left(\bar{w}_i(x) - w^{-1}\right), \qquad \sum_{i=1}^{n+1} \bar{a}_i = 0, \qquad i = 1, \dots, n+1, \end{aligned} \tag{23}$$

where a_i, $\bar{a}_i$ are arbitrary constants, subject to conditions $\sum_{i=1}^{n+1} a_i = 0$, $\sum_{i=1}^{n+1} \bar{a}_i = 0$ which ensure the absence of logarithmic singularities at infinity. For the introduced ansatz (23) we claim the following result:

Proposition 4. *Let us generalize the evolution operators to be as follows:*

$$\begin{aligned} &\mathcal{H}_0 = r(x)w + \tfrac{1}{2}u(x), \qquad \bar{\mathcal{H}}_0 = \bar{r}(x)w^{-1} + \tfrac{1}{2}\bar{u}(x), \\ &\mathcal{H}_i = \log(w_i(x) - w) + \tfrac{1}{2}\log(r(x)/w_i(x)), \\ &\bar{\mathcal{H}}_i = \log\left(\bar{w}_i(x) - w^{-1}\right) + \tfrac{1}{2}\log(r(x)/\bar{w}_i(x)), \qquad i = 1, \dots, n+1. \end{aligned} \tag{24}$$

Then, the $2n+4$ flows generated by the Lax equations

$$\begin{aligned} &\partial_{\tau_i} z = \{\mathcal{H}_i, z\}, \qquad \partial_{\bar{\tau}_i} z = \left\{\bar{\mathcal{H}}_i, z\right\}, \\ &\partial_{\tau_i} \bar{z} = \{\mathcal{H}_i, \bar{z}\}, \qquad \partial_{\bar{\tau}_i} \bar{z} = \left\{\bar{\mathcal{H}}_i, \bar{z}\right\}, \qquad i = 0, \dots, n+1 \end{aligned} \tag{25}$$

commute. They preserve the logarithmic ansatz (23) *provided the string equation* (6) *holds.*

In other words, the 2dToda flows are tangent to the manifold of logarithmic functions if the string condition is imposed and we again have $2n+4$ flows leaving invariant a $2n+5$ dimensional sub-manifold of the 2dToda system.

It is easier to prove the proposition in the Takasaki gauge. For this purpose we give the transition formulas between different gauges before the proof. For the general 2dToda hierarchy, the Lax functions $z^{(g)}$ and $\bar{z}^{(g)}$ in an arbitrary g-gauge are expressed through the Lax–Sato gauge which corresponds to $g = 0$, $z = z^{(0)}$, $\bar{z} = \bar{z}^{(0)}$ as follows

$$\begin{aligned} z^{(g)}(w) &= z\big(w/r^{2g}\big) = r(x)^{1-2g}w + \sum_{i=-\infty}^{0} u_i^{(g)}(x)w^i, \\ \bar{z}^{(g)}\big(w^{-1}\big) &= \bar{z}\big(r^{2g}/w\big) = r(x)^{1+2g}w + \sum_{i=0}^{\infty} \bar{u}_i^{(g)}(x)w^i \end{aligned}$$

while, (omitting evident subscripts)

$$\begin{aligned} H^{(g)}(w) &= B^{(g)}(w) - \left(\tfrac{1}{2} - g\right) B^{(g)}(w = 0), \\ \bar{H}^{(g)}(1/w) &= \bar{B}^{(g)}(1/w) - \left(\tfrac{1}{2} + g\right) \bar{B}^{(g)}(1/w = 0). \end{aligned} \tag{26}$$

Here, $B^{(g)}$ and $\bar{B}^{(g)}$ are specified similarly to those in (19). In particular, in the Takasaki gauge $g = 1/2$ (below we omit superscripts (1/2))

$$z = w + u(x) + \sum_{i=1}^{n+1} a_i \log(w_i(x) - w),$$

$$\bar{z} = r(x)^2 w^{-1} + \bar{u}(x) + \sum_{i=1}^{n+1} \bar{a}_i \log\left(\bar{w}_i(x) - w^{-1}\right), \tag{27}$$

$$\mathcal{H}_0 = w + u(x), \qquad \bar{\mathcal{H}}_0 = r(x)^2 w^{-1},$$
$$\mathcal{H}_i = \log(w_i(x) - w), \qquad \bar{\mathcal{H}}_i = \log\left(\bar{w}_i(x) - w^{-1}\right) + \log(w_i(x)/r(x)). \tag{28}$$

Proof of Proposition 4.

1. *Formal commutativity of flows.* We demonstrate the commutativity of $\partial_{\tau_i} z$, and $\partial_{\tau_j} z$. The proof is similar for the rest of the flows.

The commutativity of the flows is equivalent to the zero-curvature equation

$$\{\mathcal{H}_i, \mathcal{H}_j\} - \frac{\partial \mathcal{H}_j}{\partial \tau_i} + \frac{\partial \mathcal{H}_i}{\partial \tau_j} = 0. \tag{29}$$

Equations (28), (27) and (25) imply that

$$\frac{\partial \mathcal{H}_j}{\partial \tau_i} = \frac{\partial_{\tau_i} w_j}{w_j - w} = \frac{1}{a_j}(\partial_{\tau_i} z)_j = \frac{1}{a_j}\{\mathcal{H}_i, z\}_j,$$

where subscript j stands for the singular part of the expansion around w_j (see (15)). Thus instead of the lhs of (29) we have

$$\{\mathcal{H}_i, \mathcal{H}_j\} - \frac{1}{a_j}\{\mathcal{H}_i, \mathcal{H}_j\}_j + \frac{1}{a_i}\{\mathcal{H}_i, \mathcal{H}_j\}_i$$

which is zero by direct calculation applying (28) and (5). This proves the formal commutativity of the flows.

2. *Consistency of* (27) *with equations of motion* (25). In a way similar to that of the polynomial and rational cases the consistency of the half-flows $\partial_{\tau_i} z$, and $\partial_{\bar{\tau}_i} \bar{z}$ for logarithmic solutions is automatic and does not require any extra restrictions on the coefficients (the proof is also similar). In the logarithmic ansatz the main obstacle appears for the flows $\partial_{\tau_i} \bar{z}$ and $\partial_{\bar{\tau}_i} z$. We prove the consistency for $\partial_{\bar{\tau}_i} z$. The proof for $\partial_{\tau_i} \bar{z}$ is similar.

Differentiating z directly with respect to $\bar{\tau}_i$ and using the equation of motion (25) we get

$$\partial_{\bar{\tau}_i} z = \partial_{\bar{\tau}_i} u + \sum_{j=1}^{n+1} \frac{a_i \partial_{\bar{\tau}_i} w_j}{w - w_j} = \{\bar{\mathcal{H}}_i, z\}.$$

The lhs of the equation contains only singularities of z. Using definition (5) and equations (27), (28), we see that the rhs contains both singularities of z and one singularity of $\bar{\mathcal{H}}_i$. In order for the equation of motion to be consistent with the logarithmic ansatz, the rhs, $\{\bar{\mathcal{H}}_i, z\}$, must contain the same singularities as those in the lhs, i.e. it must not contain a singularity of $\bar{\mathcal{H}}_i$ at $w = 1/\bar{w}_i$.

Let us exploit the fact that in the Takasaki gauge (27), (28), z and $\bar{z}$ can be represented as sums over the corresponding $\mathcal{H}$:

$$z = \mathcal{H}_0 + \sum_{i=1}^{n+1} a_i \mathcal{H}_i, \qquad \bar{z} = \bar{\mathcal{H}}_0 + \sum_{i=1}^{n+1} \bar{a}_i \bar{\mathcal{H}}_i + f(r, w_1, \ldots, w_{n+1}), \tag{30}$$

where f is a w-independent function. Note that the conditions $\sum_{i=1}^{n+1} a_i = 0$, $\sum_{i=1}^{n+1} \bar{a}_i = 0$ of Proposition 4 ensure that there are no logarithmic singularities at infinity.

Now, using (30) we obtain the following expression

$$\bar{a}_i\{\bar{\mathcal{H}}_i, z\} = \Big\{\bar{z} - f - \sum_{j\neq i} \bar{a}_j\bar{\mathcal{H}}_j, z\Big\} = \{\bar{z}, z\} - \{f, z\} - \sum_{j\neq i} \bar{a}_j\{\bar{\mathcal{H}}_j, z\}.$$

The term $\sum\limits_{j\neq i} \bar{a}_j\{\bar{\mathcal{H}}_j, z\}$ of the rhs of the last expression does not contain singularities of $\bar{H}_i$ since it does not contain the index i in the sum. The next term $\{f, z\}$ does not contain them either, since f is independent of w. The only term which may contain undesired singularities is $\{\bar{z}, z\}$, however the string equation (6) holds and therefore $\{\bar{\mathcal{H}}_i, z\}$ is free of singularities of $\bar{z}$. Thus the equation of motion is consistent with the logarithmic ansatz due to the string equation. ■

Corollary 1. *The consistency of the rational ansatz follows from the consistency of the logarithmic ansatz. (Proposition 3 is a consequence of Proposition 4).*

Proof. Choosing $n = 2N - 1$ and substituting the values

$$\begin{aligned} &a_{2i-1} = 1/\epsilon, \qquad a_{2i} = -1/\epsilon, \qquad w_{2i} = w_{2i-1} + \epsilon u_i,\\ &\bar{a}_{2i-1} = 1/\epsilon, \qquad \bar{a}_{2i} = -1/\epsilon, \qquad \bar{w}_{2i} = \bar{w}_{2i-1} + \epsilon \bar{u}_i \end{aligned} \tag{31}$$

for z and $\bar{z}$ into (23) we get rational solutions (16), (13) in the $\epsilon \to 0$ limit.

The Hamiltonians are then related as follows:

$$\begin{aligned} &h_0 = \lim_{\epsilon=0} \mathcal{H}_0, && \bar{h}_0 = \lim_{\epsilon=0} \bar{\mathcal{H}}_0,\\ &h_{2i-1} = \lim_{\epsilon=0} \frac{1}{\epsilon}(\mathcal{H}_{2i} - \mathcal{H}_{2i-1}), && \bar{h}_{2i} = \lim_{\epsilon=0} \frac{1}{\epsilon}(\bar{\mathcal{H}}_{2i} - \bar{\mathcal{H}}_{2i-1}),\\ &h_{2i} = \lim_{\epsilon=0} \mathcal{H}_{2i-1}, && \bar{h}_{2i-1} = \lim_{\epsilon=0} \bar{\mathcal{H}}_{2i-1}. \end{aligned}$$

■

Thus we have obtained a rational ansatz as a limiting case of the logarithmic ansatz, merging pairs of logarithmic singularities together, in which case they become simple poles. In a similar way one may deduce any kind of rational maps containing a combination of poles of any degrees absorbing different numbers of logarithmic singularities.

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