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2-Dimensional Plasticity: Boundary Problems and Conservation Laws, Reproduction of Solutions

Sergey I. SENASHOV † *and Alexander YAKHNO* ‡

† *Krasnoyarsk State Trade-Economic Institute, 2 Prushinskoi Str., 660075 Krasnoyarsk, Russia*
E-mail: *senashov@kgtei.kts.ru*

‡ *University of Guadalajara, C.U.C.E.I., Guadalajara, Mexico*
E-mail: *ay223627@cucei.udg.mx*

For the system of two-dimensional plasticity equations with the help of conservation laws basic boundary problems are solved. We have considered solutions of some concrete boundary problems. Using the symmetry transformations, from the well known Prandtle's solution we deduce some new solutions, describing the pressing of a layer between two rugged slabs. Also, some new properties of characteristics of the plane plasticity are shown.

1 Introduction

Among hyperbolic systems of nonlinear equations with partial derivatives the systems of the quasilinear equations of two independent variables are most investigated. These systems describe, in particular, unsteady one-dimensional and supersonic two-dimensional stationary flows of compressible gases and liquids, two-dimensional deformed plastic state of a continuous medium and so on. A lot of them are reduced to a hyperbolic system of homogeneous quasilinear equations

$$\begin{aligned} &u_x + A(u,v)u_y = 0, \\ &v_x + B(u,v)v_y = 0, \end{aligned} \tag{1}$$

where $u = u(x,y)$, $v = v(x,y)$, an indices below mean derivation with respect to the corresponding variables.

In the Section 2 we describe a method of the analytical solving of boundary-value problems for the system (1). This method is based on the conservations laws that allow us to consider linearization of problems without nonsingular transformations and to obtain exact solutions of boundary-value problems in explicit form. In the Section 3 we consider some concrete systems.

Other way to solve some boundary-value problems is using of symmetries of equations. The theory of symmetries is widely used in investigation and in resolution of differential equations. A group of symmetries allows to find exact solutions of differential equations and to classify PDE's by admitting groups.

Symmetries admitted by the system of PDE's have a great property: under its action any solutions of the system are transformed to the solutions of this system. This property allows to construct new solutions without integrating of the given system, only by means of group transformations of known solutions. In such a way some of interesting results were obtained [1]. Note that this way is effective only if we have a sufficiently rich group of point transformations. In the Section 4 Prandtl's solution was transformed. As a result we obtained a lot of exact solutions. We selected only restricted ones along Oy-axis ones and these solutions can be used for the solving of boundary-value problems for compression of a plastic layer by rigid parallel slabs. We can hope that with the help of these solutions it will be possible to describe not only compression of the thin layers, but the thick ones too.

2 Conservation laws

Let us set Cauchy's problem: in some neighborhood of an arc $a \le \tau \le b$ of a smooth curve

$$L = \{(x,y) : x = x(\tau), y = y(\tau), \tau \in [a,b]\}$$

in the plane xOy it is required to find a solution of the system (1) that takes given values on L

$$u(x,y)|_L = u(x(\tau), y(\tau)) = u^0(\tau), \qquad v(x,y)|_L = v(x(\tau), y(\tau)) = v^0(\tau).$$

The characteristics equations of the system (1) look like

$$\frac{dx}{dy} = A, \qquad \frac{dx}{dy} = B, \tag{2}$$

with the relations on the characteristics $u = u^0$, $v = v^0$ respectively.

We search a conservation law of the set of equations (1) in the form of a relation

$$C_x + D_y = 0, \tag{3}$$

where $C = C(u,v)$, $D = D(u,v)$, which should vanish only on the solutions of the system (1):

$$C_u u_x + C_v v_x + D_u u_y + D_v v_y = -AC_u u_y - BC_v v_y + D_u u_y + D_v v_y = 0,$$

hence

$$D_u - AC_u = 0, \qquad D_v - BC_v = 0. \tag{4}$$

The equation (3), if conditions of Green's theorem are satisfied, is equivalent to a relation

$$\oint_\Gamma -C dy + D dx = 0,$$

where Γ is an arbitrary closed contour.

In the plane xOy we will consider the closed path MNK, where $M(x_m(a), y_m(a))$, $N(x_n(b), y_n(b)) \in L$, $K(x_k, y_k)$ is the point of an intersection of characteristics $v = v^0$, $u = u^0$, drawn through the points M, N respectively.

Then,

$$\oint_{MNK} -C dy + D dx = \int_{MN} D dx - C dy + \int_{NK} D dx - C dy + \int_{KM} D dx - C dy = 0. \tag{5}$$

Taking into account the expressions (2)

$$\int_{NK} D dx - C dy = \int_{NK} (D - AC) dx = x(D - AC)\big|_{x_n}^{x_k} - \int_{NK} x \partial_v (D - AC) dv.$$

Similarly,

$$\int_{KM} D dx - C dy = \int_{KM} (D - BC) dx = x(D - BC)\big|_{x_k}^{x_m} - \int_{KM} x \partial_u (D - BC) du.$$

Let us assume

$$\partial_v \left[(D - AC)|_{u=u^0(x(b),y(b))} \right] = 0, \qquad \partial_u \left[(D - BC)|_{v=v^0(x(a),y(a))} \right] = 0. \tag{6}$$

Let us denote

$$\phi(u,v) = D - AC, \qquad \psi(u,v) = D - BC.$$

Then

$$D = (A\psi - B\phi)/(A - B), \qquad C = (\psi - \phi)/(A - B),$$

where $A \neq B$, because the considered system (1) is the hyperbolic one and has two different families of characteristics.

In the new variables, the system (4) has the form

$$\begin{aligned} &\phi_u + K_1(\psi - \phi) = 0, \\ &\psi_v + K_2(\psi - \phi) = 0, \end{aligned} \tag{7}$$

where $K_1 = A_u/(A - B)$, $K_2 = B_v/(A - B)$.

We will write the conditions (6) as follows

$$\phi|_{u=u^0} = \text{const}_1 = 1, \qquad \psi|_{v=v^0} = \text{const}_2 = 0. \tag{8}$$

Coming back to (5) and taking into account (8), we obtain

$$\begin{aligned} \int_{MN} Ddx - Cdy &= -(x(D - AC)|_{x_n}^{x_k} + x(D - BC)|_{x_k}^{x_m}) \\ &= -(x_k\phi|_{u=u^0,v=v^0} - x_n\phi|_{u=u^0} + x_m\psi|_{v=v^0} - x_k\psi|_{u=u^0,v=v^0}) = x_n - x_k. \end{aligned} \tag{9}$$

If we find a solution of the linear system (7) satisfying to the boundary conditions (8), then we will determine the coordinate x_k from the equation (9).

On the other hand, for y-coordinate

$$\int_{NK} Ddx - Cdy = \int_{NK} \left(\frac{D}{A} - C\right) dy = y\,\frac{\phi}{A}\bigg|_{y_n}^{y_k} - \int_{NK} y\partial_v\left(\frac{\phi}{A}\right) dv.$$

Similarly,

$$\int_{KM} Ddx - Cdy = \int_{KM} \left(\frac{D}{B} - C\right) dy = y\,\frac{\psi}{B}\bigg|_{y_k}^{y_m} - \int_{KM} y\partial_u\left(\frac{\psi}{B}\right) du.$$

Let us assume

$$\partial_v\left[\frac{\phi}{A}\bigg|_{u=u^0(x(b),y(b))}\right] = 0, \qquad \partial_u\left[\frac{\psi}{B}\bigg|_{v=v^0(x(a),y(a))}\right] = 0. \tag{10}$$

We can write the conditions (10) as follows

$$\phi|_{u=u^0} = A(u^0, v), \qquad \psi|_{v=v^0} = 0. \tag{11}$$

From (5), taking into account (10), we have

$$\begin{aligned} \int_{MN} Ddx - Cdy = &-y_k\,\frac{\phi}{A}\bigg|_{u=u^0,v=v^0} + y_n\,\frac{\phi}{A}\bigg|_{u=u^0} \\ &- y_m\,\frac{\psi}{B}\bigg|_{v=v^0} + y_k\,\frac{\psi}{B}\bigg|_{u=u^0,v=v^0} = y_n - y_k. \end{aligned} \tag{12}$$

The solution of the problem (7), (11) makes possible to find the coordinate y_k from the equation (12). Thus, we will determine the coordinates of the point K, where the values of the functions u, v can be reconstructed.

3 Examples of application of the method

1. In the work [2] the solution was found that takes given values on L

$$\sigma(x,y)|_L = \sigma_0, \qquad \theta(x,y)|_L = \theta_0$$

for the initial problem for the system of 2-dimensional plasticity under von Mises condition

$$\begin{aligned} &\sigma_x - 2k\left(\theta_x \cos 2\theta + \theta_y \sin 2\theta\right) = 0,\\ &\sigma_y - 2k\left(\theta_x \sin 2\theta - \theta_y \cos 2\theta\right) = 0, \end{aligned} \tag{13}$$

where σ is a hydrostatic pressure, θ is an angle between the first main direction of a stress tensor and the ox-axis, k is a constant of plasticity. This system in the form (1) is

$$\xi_x + \xi_y \tan\theta = 0, \qquad \eta_x - \eta_y \cot\theta = 0,$$

where $2\theta = \eta - \xi$, $\sigma = k(\eta + \xi)$. Solution of problem (7), (8) has the form: $\phi = \rho/\cos\theta$, $\psi = 2\rho_\xi/\sin\theta$, where

$$\rho(\xi,\eta) = R(\xi,\xi_0,\eta,\eta_0)\cos\left(\frac{\eta_0 - \xi_0}{2}\right) - \frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\sin\left(\frac{\tau - \xi_0}{2}\right) d\tau.$$

Accordingly, the solution of the problem (7), (11) is

$$\rho(\xi,\eta) = R(\xi,\xi_0,\eta,\eta_0)\sin\left(\frac{\eta_0 - \xi_0}{2}\right) + \frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\cos\left(\frac{\tau - \xi_0}{2}\right) d\tau,$$

where $R(\xi,\eta,\xi_0,\eta_0) = I_0\left(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}\right)$ is Bessel function of a zero order of imaginary argument, $I_0(0) = 1$, ${I_0}'(0) = 0$.

2. The generalization of (13) is a system of ideal plasticity equations under Coulomb's plasticity condition that has the form:

$$\begin{aligned} &\sigma_x(1 + \cos 2\alpha\cos 2\Theta) + \sigma_y \cos 2\alpha \sin 2\Theta = 2(\sigma\cos 2\alpha + k\sin 2\alpha)(\Theta_x \sin 2\Theta - \Theta_y \cos 2\Theta),\\ &\sigma_x \cos 2\alpha \sin 2\Theta + \sigma_y(1 - \cos 2\alpha\cos 2\Theta) = -2(\sigma\cos 2\alpha + k\sin 2\alpha)(\Theta_x \cos 2\Theta + \Theta_y \sin 2\Theta), \end{aligned}$$

where $\pi/2 - 2\alpha$ is a constant angle of internal friction, $\Theta = \theta + \pi/4$. If $\alpha = \pi/4$, then we have the system (13). This system in the form (1) is:

$$\xi_x + \xi_y \tan(\Theta - \alpha) = 0, \qquad \eta_x - \eta_y \tan(\Theta + \alpha) = 0,$$

where $\xi = \frac{1}{2}\tan 2\alpha \ln(\sigma\cot 2\alpha + k) - \Theta$, $\eta = \frac{1}{2}\tan 2\alpha \ln(\sigma\cot 2\alpha + k) + \Theta$. Solution of problem (7), (8) has a form [3]: $\phi = \gamma(-\xi,-\eta)V(\xi,\eta)/\cos(\Theta-\alpha)$, $\gamma(\xi,\eta) = \exp(-(\xi+\eta)/2\cot 2\alpha)$, where:

$$\begin{aligned} V = {} & \gamma(\xi_0,\eta_0)R(\xi,\xi_0,\eta,\eta_0)\cos\left((\eta_0 - \xi_0)/2 - \alpha\right)\\ & - \frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\gamma(\xi_0,\tau)\left[\sin\left((\tau-\xi_0)/2 - \alpha\right) - \cot 2\alpha\cos\left((\tau - \xi_0)/2 - \alpha\right)\right] d\tau. \end{aligned} \tag{14}$$

The solution of the problem (7), (11) is

$$\begin{aligned} V = {} & \gamma(\xi_0,\eta_0)R(\xi,\xi_0,\eta,\eta_0)\sin\left((\eta_0 - \xi_0)/2 - \alpha\right)\\ & - \frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\gamma(\xi_0,\tau)\left[\cot 2\alpha\sin\left((\tau-\xi_0)/2 - \alpha\right) - \cos\left((\tau - \xi_0)/2 - \alpha\right)\right] d\tau, \end{aligned} \tag{15}$$

$R(\xi,\eta,\xi_0,\eta_0) = I_0\left(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}/\sin 2\alpha\right)$. From the equation (9) taking into account (14) we will find a coordinate x_k. Using (15), from the equations (12) we will obtain a coordinate y_k. Thus, we have determined a point K, in which the values of functions ξ, η are reconstructed.

4 Reproduction of Prandle's solution

Let us consider the system (13) that has two families of characteristics (in the theory of plasticity they are named slide lines) with the following equations and relations along characteristics:

$$\frac{dy}{dx} = \tan\theta, \qquad \frac{\sigma}{2k} - \theta = c_1,$$
$$\frac{dy}{dx} = -\cot\theta, \qquad \frac{\sigma}{2k} + \theta = c_2.$$

This system admits the infinite group of symmetries [4]. Its point subgroup is generated by following generators:

$$X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = x\partial_x + y\partial_y, \qquad X_4 = -x\partial_y + y\partial_x + \partial_\theta,$$
$$X_5 = \partial_\sigma, \qquad X_6 = \xi_1\partial_x + \xi_2\partial_y + 4k\theta\partial_\sigma - \frac{\sigma}{k}\partial_\theta, \qquad X = \xi\partial_x + \eta\partial_y, \tag{16}$$

where

$$\xi_1 = -x\cos 2\theta - y\sin 2\theta - y\frac{\sigma}{k}, \qquad \xi_2 = y\cos 2\theta - x\sin 2\theta + x\frac{\sigma}{k},$$

and (ξ, η) is an arbitrary solution of the following lineal system of equations:

$$\xi_\theta - 2k\left(\xi_\sigma\cos 2\theta + \eta_\sigma\sin 2\theta\right) = 0,$$
$$\eta_\theta - 2k\left(\xi_\sigma\sin 2\theta - \eta_\sigma\cos 2\theta\right) = 0, \tag{17}$$

Transformations, which correspond to any generator of (16), convert the system (13) to itself. Thus, for the generator X we have transformations of independent variables:

$$x' = x + a\xi, \qquad y' = y + a\eta, \tag{18}$$

where a is an arbitrary parameter, (ξ, η) is an arbitrary solution of (17). As X is an admitted generator, the transformations (18) do not change the form of the system (13). Note that we have shown the variables that are changed under this transformation, other variables are omitted.

Under the action of the symmetry group, any solution of the system (13) is transformed to the solution of this system. Furthermore, it is easy to demonstrate

Proposition 1. *The characteristics of the system* (13) *are transformed to the characteristics of this system under the action of symmetry group* (16)*.*

The system (13) is being investigated for more than 100 years, but there are a few of its exact solutions. Certainly, one of the well-known solutions was constructed by Prandtl [5]. Let us write this solution in the form

$$\sigma = -kx + k\left(1 - y^2\right)^{\frac{1}{2}}, \qquad 2\theta = \arccos y. \tag{19}$$

From proposition, in particular, it is follows that for the construction of slide lines we should substitute the characteristic relations to the solution. Let us substitute the relation along the first characteristic $\sigma/(2k) - \theta = c_1$ to the solution (19). We obtain the equation of the first family of slide lines:

$$x = -2\theta + \sin 2\theta + K_1, \qquad y = \cos 2\theta.$$

By analogy, we obtain the second family of slide lines with relation $\sigma/(2k) + \theta = c_2$:

$$x = 2\theta + \sin 2\theta + K_2,\ \ y = \cos 2\theta.$$

Under the transformations (18), Prandtl's solution will transform to the following one:

$$\sigma = -kx + ak\xi + k\sin 2\theta, \qquad y = \cos 2\theta + a\eta, \tag{20}$$

and the slide lines will have the form:

$$x = \mp 2\theta + a\xi + \sin 2\theta + K_i, \qquad y = \cos 2\theta + a\eta, \qquad i = 1, 2.$$

Therefore, in order to obtain new solutions, we have to find some solutions of the system (17). This is the system of PDE's of the first order with the variable coefficients, and it is possible to find solutions in the form of series:

$$\xi = \sum_{i=1}^{\infty} \left[f_i(\sigma)\cos(i\theta) + g_i(\sigma)\sin(i\theta) \right], \qquad \eta = \sum_{i=1}^{\infty} \left[F_i(\sigma)\cos(i\theta) + G_i(\sigma)\sin(i\theta) \right],$$

where $f_i(\sigma)$, $g_i(\sigma)$, $F_i(\sigma)$, $G_i(\sigma)$ are the functions, obtained as solutions of a system of ODE's. This way can be successful, but it is difficult to give an analysis of results and use them for practical goals. Therefore, we can find some solutions of (17) in a simpler way, which can be used in practice. Solutions of the system (17) we can find in the forms:

$$\xi = \alpha\sigma + F(\theta), \qquad \eta = \beta\sigma + G(\theta), \tag{21}$$

$$\xi = f(\theta)\exp\left(\frac{\sigma}{2k}\right), \qquad \eta = g(\theta)\exp\left(\frac{\sigma}{2k}\right), \tag{22}$$

$$\xi = \alpha(\theta)\sin\frac{\sigma}{2k} + \beta(\theta)\cos\frac{\sigma}{2k}, \qquad \eta = F(\theta)\sin\frac{\sigma}{2k} + G(\theta)\cos\frac{\sigma}{2k}, \tag{23}$$

$$\xi = \alpha(\theta)\sinh\frac{\sigma}{2k} + \beta(\theta)\cosh\frac{\sigma}{2k}, \qquad \eta = F(\theta)\sinh\frac{\sigma}{2k} + G(\theta)\cosh\frac{\sigma}{2k}, \tag{24}$$

where functions f, g, α, β, F, G are determined as a solution of corresponding systems of ODEs.

It is easy to see that under the transformation (21) the Prandtl's solutions is transformed into the solution of the same form (there is no new solution).

Under the transformations (22), (24) the solution will be transformed to the non-restricted ones along the oy-axis, hence we do not consider solutions of this form in this article.

Let us consider the action of ultimate transformation (23) to the slide lines. Substituting (23) to the system (17) we have the system of equations for a determination of functions α, β, F, G

$$\begin{aligned} &\alpha' + \beta\cos 2\theta + G\sin 2\theta = 0, \qquad \beta' - \alpha\cos 2\theta - F\sin 2\theta = 0, \\ &F' + \beta\sin 2\theta - G\cos 2\theta = 0, \qquad G' - \alpha\sin 2\theta + F\cos 2\theta = 0. \end{aligned} \tag{25}$$

Let us derive each of the equations (25) with respect to θ. Then, according to (25), we obtain the equations:

$$\begin{aligned} &\alpha'' + \alpha + 2F' = 0, \qquad F'' + F - 2\alpha' = 0, \\ &\beta'' + \beta + 2G' = 0, \qquad G'' + G - 2\beta' = 0, \end{aligned} \tag{26}$$

Let us find the solution of the first system of (26) in the form

$$\alpha = Ce^{\lambda\theta}, \qquad F = De^{\lambda\theta}, \qquad C, D = \text{const}.$$

Therefore, for the determination of λ, C, D we have the algebraic equations

$$C\lambda^2 + C + 2D\lambda = 0, \qquad D\lambda^2 + D - 2C\lambda = 0.$$

Hence $C = \pm iD$, $\lambda_{1,2} = i(1 \pm \sqrt{2})$, $i^2 = -1$. Then, a real solution for α, F has the form

$$\alpha = C_1 \cos\left(1 \pm \sqrt{2}\right)\theta - C_2 \sin\left(1 \pm \sqrt{2}\right)\theta,$$
$$F = C_1 \sin\left(1 \pm \sqrt{2}\right)\theta + C_2 \cos\left(1 \pm \sqrt{2}\right)\theta,$$

where C_1, C_2 are arbitrary constants.

Then, solution for β, G of the second system (26), according to (25) will take the form:

$$\beta = \left(1 \pm \sqrt{2}\right)\left(C_2 \cos\left(1 \mp \sqrt{2}\right)\theta - C_1 \sin\left(1 \mp \sqrt{2}\right)\theta\right),$$
$$G = \left(1 \pm \sqrt{2}\right)\left(C_2 \sin\left(1 \mp \sqrt{2}\right)\theta + C_1 \cos\left(1 \mp \sqrt{2}\right)\theta\right)$$

For a simplicity let us $C_2 = 0$, and let us take the lower sign, then (23) we can write down in the form:

$$\xi = \cos\left(1 - \sqrt{2}\right)\theta \sin\frac{\sigma}{2k} - \left(1 - \sqrt{2}\right)\sin\left(1 + \sqrt{2}\right)\theta \cos\frac{\sigma}{2k},$$
$$\eta = \sin\left(1 - \sqrt{2}\right)\theta \sin\frac{\sigma}{2k} + \left(1 - \sqrt{2}\right)\cos\left(1 + \sqrt{2}\right)\theta \cos\frac{\sigma}{2k}.$$

The transformations of the slide lines can be written as:

$$\begin{aligned} x &= \mp 2\theta + a\left[\cos\left(1 - \sqrt{2}\right)\theta \sin(\pm\theta + c_i) - \left(1 - \sqrt{2}\right)\sin\left(1 + \sqrt{2}\right)\theta \cos(\pm\theta + c_i)\right] \\ &\quad + \sin 2\theta + K_i, \qquad i = 1, 2, \\ y &= \cos 2\theta + a\left[\sin\left(1 - \sqrt{2}\right)\theta \sin(\pm\theta + c_i) + \left(1 - \sqrt{2}\right)\cos\left(1 + \sqrt{2}\right)\theta \cos(\pm\theta + c_i)\right]. \end{aligned} \tag{27}$$

From relations (27) it follows, that new solution has restricted slide lines, and we can use these solutions for description of plastic flows between two rugged slabs.

With the increment of parameter a the thickness of layer increases and becomes approximately equal to $2(h + a)$, where h is a thickness of the initial layer. The new solution that we named S-solution, i.e. obtained from the Prandtl's solution by means of symmetry transformations (20), has the form:

$$\sigma = -kx + a\left[\cos\left(1 - \sqrt{2}\right)\theta \sin\frac{\sigma}{2k} - \left(1 - \sqrt{2}\right)\sin\left(1 + \sqrt{2}\right)\theta \cos\frac{\sigma}{2k}\right] + k\left(1 - y^2\right)^{\frac{1}{2}},$$
$$y = \cos 2\theta + a\left[\sin\left(1 - \sqrt{2}\right)\theta \sin\frac{\sigma}{2k} + \left(1 - \sqrt{2}\right)\cos\left(1 + \sqrt{2}\right)\theta \cos\frac{\sigma}{2k}\right],$$

where a is an arbitrary parameter. For large values of parameter a we have S-solution far different from Prandle's solution. It can be used for analysis of not thin plastic flows.

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