Generally Covariant Quantum Mechanics

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We present a complete theory, which is a generalization of Bargmann's theory of factors for ray representations. We apply the theory to the generally covariant formulation of the Quantum Mechanics.

1 Problem

In the standard Quantum Mechanics (QM) and the Quantum Field Theory (QFT) the space-time coordinates are pretty classical variables. Therefore the question about the general covariance of QM and QFT emerges naturally just like in the classical theory:

what is the effect of a change in space-time coordinates in QM and QFT when the change does not form any symmetry transformation?

It is a commonly accepted belief that there are no substantial difficulties if we refer the question to the wave equation. We simply treat the wave equation, and do not say why, in such a manner as if it was a classical equation. The only problem arising is to find the transformation rule $\psi \to T_r\psi$ for the wave function ψ. This procedure, which on the other hand can be seriously objected, does not solve the above-stated problem. The heart of the problem as well as that of QM and QFT, lies in the Hilbert space of states, and specifically in finding the representation T_r of the covariance group in question. The trouble gets its source in the fact that the covariance transformation changes the form of the wave equation such that ψ and $T_r\psi$ do not belong to the same Hilbert space, which means that T_r does not act in the ordinary Hilbert space. This is not compatible with the paradigm worked out in dealing with symmetry groups.

2 Resume of the reformulation of the Quantum Mechanics

We have shown that covariance group acts in a Hilbert bundle $\mathcal{R}\triangle\mathcal{H}$ over time in the non-relativistic theory and in a Hilbert bundle $\mathcal{M}\triangle\mathcal{H}$ over space-time $\mathcal{M}$ in the relativistic case. The wave functions are the appropriate cross sections of the bundle in question. The exponent $\xi(r, s, p)$ in the formula

$$T_r T_s = e^{i\xi(r,s,p)} T_{rs},$$

depends on the point p of the base of the bundle in question: that is, ξ depends on the time t in the nonrelativistic theory and on space-time point p in the relativistic theory if there exists a nontrivial gauge freedom.

Moreover, we argue that the bundle $\mathcal{M}\triangle\mathcal{H}$ is more appropriate for treating the covariance as well as the symmetry groups then the Hilbert space itself. Namely, we show that from the more general assumption that the representation T_r of the Galilean group acts in $\mathcal{R}\triangle\mathcal{H}$ and has an exponent $\xi(r, s, t)$ depending on the time t we reconstruct the nonrelativistic Quantum Mechanics. Even more, in the less trivial case of the theory with nontrivial time-dependent

gauge describing a spin less quantum particle in the Newtonian gravity we are able to infer the wave equation and prove the equality of the inertial and gravitational masses.

In doing it we apply extensively the classification theory for exponents $\xi(r, s, t)$ of T_r acting in $\mathcal{R}\triangle\mathcal{H}$ and depending on the time.

In the presented theory which is slightly more general then the standard one the gauge freedom emerges from the very nature of the fundamental laws of Quantum Mechanics. By this it opens a new perspective in solving the troubles in QFT caused by the gauge freedom.

Interpretation. The physical interpretation ascribed to the cross-section ψ is as follows. Each experiment is, out of its very nature, a spatiotemporal event. To each act of measurement carried out at the space-time point p_0 we ascribe a self-adjoint operator Q_{p_0} acting in the Hilbert space $\mathcal{H}_{p_0}$ and ascribe to the spectral theorem for Q_{p_0} the standard interpretation. Hence, assuming for simplicity that Q_{p_0} is bounded, if $\phi_0 \in \mathcal{H}_{p_0}$ and $\lambda_0 = \lambda_o(p_0)$ is a characteristic vector and its corresponding characteristic value of Q_{p_0} respectively then we have the following statement. *If the experiment corresponding to Q_p was performed at the spatiotemporal event p_0 on a system in the state described by the cross section ψ, then the probability of the measurement value to be $\lambda_0(p_0)$ and the system to be found in the state described by ϕ such that $\phi(p_o) = \phi_0$ after the experiment is given by the square of the absolute value of the Borel function $|f_\psi(p_0, \phi_0)|^2 = |(\phi_0, \psi_{p_0})|^2$ induced by the cross section ψ.* In the nonrelativistic case the above statement is a mere rephrasing of the well established knowledge.

See [1] for detail treatment, where the theory was proposed.

3   Resume of the generalization of the Bargmann's theory of factors

Our main task was to construct the general classification theory of space-time dependent exponents $\xi(r, s, p)$ of representations acting in $\mathcal{M}\triangle\mathcal{H}$, see [1]. On the other hand the presented theory can be viewed as a generalization of the Bargmann's [2] classification theory of exponents $\xi(r, s)$ of representations acting in ordinary Hilbert spaces, which are independent of $p \in \mathcal{M}$.

Definition 1. By an *isomorphism* of the Hilbert bundle $\mathcal{M}\triangle\mathcal{H}$ with the Hilbert bundle $\mathcal{M}'\triangle\mathcal{H}'$ we shall mean a Borel isomorphism T of $\mathcal{M}\triangle\mathcal{H}$ on $\mathcal{M}'\triangle\mathcal{H}'$ such that for each $p \in \mathcal{M}$ the restriction of T to $p \times \mathcal{H}_p$ has some $q \times \mathcal{H}'_q$ for its range and is unitary when regarded as a map of $\mathcal{H}_p$ on $\mathcal{H}'_q$. The induced map carrying p into q is clearly a Borel isomorphism of $\mathcal{M}$ with $\mathcal{M}'$ and we denote it by T^π. The above defined T is said to be an *automorphism* if $\mathcal{M}\triangle\mathcal{H} = \mathcal{M}'\triangle\mathcal{H}'$. Note that for any automorphism T we have $(T\psi, T\phi)_{T^\pi p} = (\psi, \phi)_p$, but in general $(T\psi, T\phi)_p \neq (\psi, \phi)_p$. By this any automorphism T is what is frequently called a *bundle isometry*.

(We use the Hilbert bundle with the ordinary Borel structure in the total space and with the ordinary manifold structure in the base $\mathcal{M}$, see e.g. [3].)

Definition 2. The function $r \rightarrow T_r$ from a group G into the set of automorphisms (bundle isometries) of $\mathcal{M}\triangle\mathcal{H}$ is said to be a *general factor representation* of G associated to the action $G \times \mathcal{M} \ni r, p \rightarrow r^{-1}p \in \mathcal{M}$ of G on $\mathcal{M}$ if $T_r^\pi(p) \equiv r^{-1}p$ for all $r \in G$, and T_r satisfy the condition

$$T_r T_s = e^{i\xi(r,s,p)} T_{rs}.$$

Here we give only the summing up

Theorem 1. *(1) On a Lie group G, every local exponent $\xi(r, s, p)$ is equivalent to a canonical local exponent $\xi'(r, s, p)$ which, on some canonical neighborhood $\mathfrak{N}_0$, is analytic in canonical*

coordinates of r and s and and vanishes if r and s belong to the same one-parameter subgroup. Two canonical local exponents ξ, ξ' are equivalent if and only if $\xi'(r, s, p) = \xi(r, s, p) + \Lambda(r, p) + \Lambda(s, r^{-1}p) - \Lambda(rs, p)$ on some canonical neighborhood, where $\Lambda(r, p)$ is a linear form in the canonical coordinates of r such that $\Lambda(r, sp)$ does not depend on s if s belongs to the same one-parameter subgroup as r. (2) To every canonical local exponent of G corresponds uniquely an infinitesimal exponent $\Xi(a, b, p)$ on the Lie algebra $\mathfrak{G}$ of G, i.e. a bilinear antisymmetric form which satisfies the identity[1] $\Xi([a, a'], a'', p) + \Xi([a', a''], a, p) + \Xi([a'', a], a', p) = \mathbf{a}\Xi(a', a'', p) + \mathbf{a'}\Xi(a'', a, p) + \mathbf{a''}\Xi(a, a', p)$. The correspondence is linear. (3) Two canonical local exponents ξ, ξ' are equivalent if and only if the corresponding Ξ, Ξ' are equivalent, i.e.[2] $\Xi'(a, b, p) = \Xi(a, b, p) + \mathbf{a}\Lambda(b, p) - \mathbf{b}\Lambda(a, p) - \Lambda([a, b], p)$ where $\Lambda(a, p)$ is a linear form in a on $\mathfrak{G}$ such that $\tau \to \Lambda(a, (\tau b)p)$ is constant if $a = b$. (4) There exist a one-to-one correspondence between the equivalence classes of local exponents ξ (global in $p \in \mathcal{M}$) of G and the equivalence classes of infinitesimal exponents Ξ of $\mathfrak{G}$.

4   The physical motivation

It will be instructive to investigate the problem for a free particle in the flat Galilean space-time. The set of solutions ψ of the Schrödinger equation which are admissible in Quantum Mechanics is precisely given by

$$\psi(\vec{x}, t) = (2\pi)^{-3/2} \int \varphi(\vec{k}) e^{-i\frac{t}{2m}\vec{k}\cdot\vec{k} + i\vec{k}\cdot\vec{x}} \, d^3k,$$

where $p = \hbar k$ is the linear momentum and $\varphi(\vec{k})$ is any square integrable function. The functions φ (wave functions in the "Heisenberg picture") form a Hilbert space $\mathcal{H}$ with the inner product

$$(\varphi_1, \varphi_2) = \int \varphi_1^*(\vec{k})\varphi_2(\vec{k}) \, d^3k.$$

The correspondence between ψ and φ is one-to-one.

But in general the construction fails if the Schrödinger equation possesses a nontrivial gauge freedom. We explain it.

We need not to use the Fourier transform. **What is the role of the Schrödinger equation in the above construction of $\mathcal{H}$?** In the above construction the Hilbert space $\mathcal{H}$ is isomorphic to the space of square integrable functions $\varphi(\vec{x}) \equiv \psi(\vec{x}, 0)$ – the set of square integrable space of initial data for the Schrödinger equation. The connection between ψ and φ is given by the time evolution $U(0, t)$ operator (by the Schrödinger equation):

$$U(0, t)\varphi = \psi.$$

The correspondence between φ and ψ has all formal properties such as in the above Fourier construction. Denote the space of the initial square integrable data φ on the simultaneity hyperplane $t(X) = t$ by $\mathcal{H}_t$. The space of wave functions $\psi(\vec{x}, t) = U(0, t)\varphi(\vec{x})$ isomorphic to the Hilbert space $\mathcal{H}_0$ of φ's is called in the common "jargon" the "Schrödinger picture".

However, the connection between $\varphi(\vec{x})$ and $\psi(\vec{x}, t)$ is not unique in general, if the wave equation possesses a gauge freedom. Namely, consider the two states φ_1 and φ_2 and ask the question: when the two states are equivalent and by this indistinguishable? The answer is as

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[1] $\mathbf{a}\Xi(b, c, p)$ stands for the differential operator $d/d\tau\Xi(b, c, (\tau a)p)|_{\tau=0}$.
[2] $\tau \to \tau a$ is a one-parameter group generated by $a \in \mathfrak{G}$.

follows: they are equivalent if

$$|(\varphi_1, \varphi)| \equiv \left| \int \psi_1^*(\vec{x}, t)\psi(\vec{x}, t) \, d^3x \right| = |(\varphi_2, \varphi)|$$

$$\equiv \left| \int \psi_2^*(\vec{x}, t)\psi(\vec{x}, t) \, d^3x \right|, \tag{1}$$

for any state φ from $\mathcal{H}$, or for all $\psi = U\varphi$ (ψ_i are defined to be $= U(0, t)\varphi_i$). Substituting φ_1 and then φ_2 for φ and making use of the Schwarz's inequality one gets: $\varphi_2 = e^{i\alpha}\varphi_1$, where α is any constant[3]. The situation for ψ_1 and ψ_2 is however different. In general the condition (1) is fulfilled if

$$\boldsymbol{\psi_2 = e^{i\Lambda(t)}\psi_1}$$

and the phase factor can depend on time. Of course it has to be consistent with the wave equation, that is, together with a solution ψ to the wave equation the wave function $e^{i\Lambda(t)}\psi$ also is a solution to the appropriately gauged wave equation. *A priori* one cannot exclude the existence of such a consistent time evolution. This is not a new observation, it was noticed by J. von Neumann[4], but it seems that it has never been deeply investigated (probably because the ordinary nonrelativistic Schrödinger equation has a gauge symmetry with constant Λ). The space of waves ψ describing the system cannot be reduced in the above way to any fixed Hilbert space $\mathcal{H}_t$ with a fixed t. So, the existence of the nontrivial gauge freedom leads to the

Hypothesis. *The two waves ψ and $e^{i\Lambda(t)}\psi$ are quantum-mechanically indistinguishable.*

Moreover, we are obliged to use the whole Hilbert bundle $\mathcal{R}\triangle\mathcal{H} : t \to \mathcal{H}_t$ over the time instead of a fixed Hilbert space $\mathcal{H}_t$, with the appropriate cross sections as the waves ψ.

Consider now an action T_r of a group G in the space of waves ψ. From our analysis it follows that it is natural to replace the ordinary postulate:

Classical-like postulate. *The group G is a symmetry group if and only if the wave equation is invariant under the transformation $x' = rx$, $r \in G$ of independent variables and the transformation $\psi' = T_r\psi$ of the wave function.*

by the more appropriate alternative:

Quantum postulate. *The group G is a symmetry group if and only if the transformation $x' = rx$, $r \in G$ of independent variables and the transformation $\psi' = T_r\psi$ of the wave function transform the wave equation into a gauge-equivalent one.*

Acceptation of the **Quantum postulate** gives a new perspective for solving the two very difficult problems [1]:

 (a) **generally covariant formulation of Quantum Mechanics,**

 (b) **the troubles in the Quantum Field Theory caused by the gauge freedom.**

Moreover, with the help of the **Hypothesis** we can see that both **(a)** and **(b)** are deeply connected [1].

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[3]This gives the conception of the ray, introduced to Quantum Mechanics by H. Weyl [4]: a physical state does not correspond uniquely to a normed state $\varphi \in \mathcal{H}$, but it is uniquely described by a *ray*, two states belong to the same ray if they differ by a constant phase factor.

[4]J. von Neumann, Mathematical Principles of Quantum Mechanics, Princeton, University Press, 1955. He did not mention about the gauge freedom on that occasion. But the gauge freedom is necessary for the equivalence of ψ_1 and $\psi_2 = e^{i\Lambda(t)}\psi_1$.

5 Connection to the troubles with the gauge freedom

Now, we return to the problem **(b)**. It should be mentioned at this place that the troubles in QFT generated by the gauge freedom are of general character, and are well known. For example, there do not exist vector particles with helicity $= 1$, which is a consequence of the theory of unitary representations of the Poincaré group, as was shown by J. Łopuszański [5]. This is apparently in contradiction with the existence of vector particles with helicity $= 1$ in nature – the photon, which is connected with the electromagnetic four-vector potential. The connection of the problem with the gauge freedom is well known [5]. We omit however the difficulty if we allow the inner product in the "Hilbert space" to be not positively defined, see [6], or [7]. Due to [5], the vector potential (promoted to be an operator valued distribution in QED) cannot be a vector field, if one wants to have the inner product positively defined then together with the coordinate transformation the gauge transformation has to be applied, which breaks the vector character of the potential. Practically it means that any gauge condition which brings the theory into the canonical form such that the quantization procedure can be consequently applied (with the positively defined inner product in the Hilbert space) breaks the four-vector character of the electromagnetic potential, the Coulomb gauge condition is an example. To achieve the Poincaré symmetry of Maxwell equations with such a gauge condition (the Coulomb gauge condition for example), it is impossible to preserve the vector character of the potential – together with the coordinate transformation a well defined (by the coordinate transformation) gauge transformation f has to be applied:

$$A_\mu \to A'_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}}(A_\nu + \partial_\nu f).$$

This means that the electromagnetic potential can form a generalized ray representation T_r of the Poincaré group at most, with the space-time-dependent factor $e^{i\xi}$ if the scalar product is positively defined. One may ask: how possible is it if the Poincaré group is not only a covariance group but at the same time a symmetry group? The solution of this paradox on the grounds of the existing theory is rather obscure. We propose the following solution. The factor $e^{i\xi}$ is space-time independent for the symmetry group but under the assumption that the fundamental space describing the states of a quantum system is the ordinary Hilbert space and the **Classical-like postulate** is true. But we have presented serious objections to this assumption. Moreover, the nonrelativistic quantum theory can be reconstructed from the more general assumption about the space of quantum mechanical states saying that it compose the space of appropriate cross sections of the Hilbert bundle $\mathcal{R}\triangle\mathcal{H}$ over time $t \in \mathcal{R}$. The Schrödinger equation can be uniquely reconstructed from the generalized ray representations of the Galilean group. We also watch for a more fundamental justification of this assumption in the presumption that time is a purely classical variable in the nonrelativistic quantum mechanics or so to speak a parameter. The most general unitary representation of the locally compact commutative group of the time real line acts in a Hilbert bundle $\mathcal{R}\triangle\mathcal{H}$ over time, see Mackey [3]. So, the assumption about the "classicity" of the time t fixes the structure of space of quantum states to be a subset of cross sections of a Hilbert bundle over the time. This is the peculiar property of the Galilean group structure that the whole construction degenerates as if we were started from the ordinary ray representation over the ordinary Hilbert space and the theorem that the phase $e^{i\xi}$ is space-time independent is true in this case, but only accidentally. The generalization to the relativistic case is natural. First we postulate the space-time coordinates to be classical commutative variables, which leads to the Hilbert bundle $\mathcal{M}\triangle\mathcal{H}$ over space-time manifold $\mathcal{M}$. The factor of the representation of the Poincaré group acting in the bundle $\mathcal{M}\triangle\mathcal{H}$ has not to be a constant with respect to space-time coordinates even when it is a symmetry group. This solves the paradox.

On the other hand H. Weyl (pages 272–276 of the Dover ed. of [4]) showed that the Quantum Kinematics represented by the Heisenberg commutation relations is nothing else but a unitary

factor representation of an Abelian group. This Weyl's construction can be viewed as emerging from the gauge freedom with the constant phase (constant Λ), i.e. a special case of the Bargmann' theory, see [2]. This is just the case of the Heisenberg' analysis of commutation relations for the atom which is determined (in the first approximation) by the Coulomb field of its nuclei charge. That is, the field which is naturally connected with the constant phase gauge freedom. But the general electromagnetic interactions possess the space-time dependent gauge freedom. So, by the construction our generalization seems to be natural for treating the quantum electromagnetic interactions.

6 Connection to the troubles with the position operator

It was shown in Section 4 that our generalization with states as cross sections in a Hilbert bundle $\mathcal{M} \triangle \mathcal{H}$ over the whole space-time $\mathcal{M}$ looses physical sense in the nonrelativistic theory. In this case we have the bundle $\mathcal{R} \triangle \mathcal{H}$ over time. It should be noted that we can recover this fact from the more general point of view. Namely, any exponent $\xi(r, s, p)$ of a generalized factor representation of the Galilean group acting in $\mathcal{M} \triangle \mathcal{H}$ is equivalent to a time dependent one: $\xi(r, s, p) = \xi(r, s, t)$, such that we can restrict the whole analysis to the bundle $\mathcal{R} \triangle \mathcal{H}$ over the time. This is a peculiar property of the Galilean group. This is not the case for the Poincaré group. The time, being a classical variable, does not posses any quantum mechanical measurement operator. But, contrary to the time, the space coordinates do possess operators in the nonrelativistic theory. In the relativistic case all the space-time coordinates play the role of the time and do not possess quantum mechanical operators. This is in agreement to the well established knowledge and reflects the standard problem in construction of the position operator in the relativistic theory, compare e.g. [3] and [8].

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