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Explicitly Reducible Integrable Natural Systems with the Integral of Full Momentum

Andrij VUS and Sergiy PIDKUYKO

Franko Lviv National University, 1 Universytets'ka Str., 79000 Lviv, Ukraine
E-mail: *matmod@franko.lviv.ua*

This paper is devoted to the problem of exact reduction of natural Hamiltonian systems. The integrable Hamiltonian systems with the full collection of first integrals being polynomial in the momenta, are considered. It is proved, that the corresponding reduced (with respect to the full momentum) Hamiltonian system possesses the full collection of first integrals, which are also polynomial in the momenta.

1 Introduction

Numerous achievements in the problem of complete integrability of the finite-dimensional Hamiltonian systems are connected with the projection method, i.e. using the known first integral one can reduce the above mentioned system to system with less number of degrees of freedom. Unfortunately, the symmetry of the reduced system often possesses imperceptible symmetry, which is hard to observe. A lot of corresponding examples for this problem are considered in [1,2]. The correspondence between the full collections of involutory integrals for the initial and reduced systems is not still described. In this paper we propose the description of such correspondence for the Hamiltonian systems admitting the full collection of first integrals, which are polynomial in the momenta. This result was first considered for the case $n = 2$ and proved in [3]. We generalize it for any $n \in \mathbb{N}$. Suppose the natural Hamiltonian system with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + W(x_1, x_2, \ldots, x_n) \tag{1}$$

is integrable by Liouville and also admits the first integral (the full momentum)

$$\mathcal{P} = \sum_{i=1}^{n} p_i.$$

The following problem arises: is it possible to state that the reduced (in regard to $\mathcal{P}$) Hamiltonian system (now with $n-1$ degrees of freedom) is also integrable by Liouville? The equivalent formulation is: *does this system also admit a full collection of integrals containing the full momentum $\mathcal{P}$?* In this paper we consider the class of n-dimensional Hamiltonian natural systems with the Hamiltonian (1).

2 Principles of exact reduction

Lemma 1. *There exists the linear canonical transformation $y = Ax$, $p = A^t q$, such that in new phase coordinates (y, q) the Hamiltonian* (1) *takes the form*

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} q_i^2 + W(y_1, y_2, \ldots, y_{n-1}),$$

where the full momentum is now $\mathcal{P} = q_n$.

Proof. The proof is based on the following remark. Obviously, there exists the orthogonal matrix $A \in SL_n(\mathbb{R})$ with the last row $a_{n1} = a_{n2} = \cdots = a_{nn} = 1/\sqrt{n}$. Therefore the above-stated transformation is canonical as the expansion of the punctual transformation of the configurational variables. ■

Further on we shall denote the new canonical coordinates and momenta as (x, p) and consider the system with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + W(x_1, x_2, \ldots, x_{n-1}). \tag{2}$$

Lemma 2. *Let $F(x, p)$ be a polynomial in the momenta first integral of the system with the Hamiltonian* (2)*. Then*
1) $\frac{\partial F}{\partial p_n}$ *is also the first integral of this system;*
2) F *is a polynomial in* x_n.

Proof. Obviously, $\{\frac{\partial F}{\partial x_n}, \mathcal{H}\} = \frac{\partial}{\partial x_n}\{F, \mathcal{H}\} = 0$. As $F(x,p)$ is a polynomial in the momenta, it can be represented in the form

$$F = F_m + F_{m-1} + \cdots + F_0,$$

where the component F_k is a homogeneous polynomial of degree k in the momenta $p_1, \ldots, p_n$. Then for the upper part F_m we obtain that the Poisson bracket $\{F_m, \frac{1}{2}\sum\limits_{i=1}^{n} p_i^2\} = 0$, i.e. $\sum\limits_{i=1}^{n} \frac{\partial F}{\partial x_i} = 0$. Therefore F_m is a polynomial in the collection of its arguments

$$\{\, p_i \mid i \in \overline{1,n}\,\} \cup \{(x_i p_j - x_j p_i) \mid i, j \in \overline{1,n}\},$$

hence the degree in the momenta of the polynomial $(\frac{\partial}{\partial x_n})^{n+1} F$ is less than n. Consequently, $(\frac{\partial}{\partial x_n})^{n(n+1)/2} F = 0$. ■

Now we consider the possibility of the reduction of Hamiltonian system with the Hamiltonian (2) to the problem with $(n-1)$ degrees of freedom and the Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2}\sum_{i=1}^{n-1} p_i^2 + W(x_1,\, x_2,\, \ldots,\, x_{n-1}). \tag{3}$$

in the following sense:

1) the initial system also admits the existing of the involutory collection of first integrals in the form $\Omega = (\mathcal{H} = F_1, F_2, \ldots, F_{n-1}, p_n)$;
2) $\Omega_0 = (\mathcal{H}_0 = F_1,\, F_2, \ldots,\, F_{n-1})$ is the involutory collection of first integrals for the reduced system with the Hamiltonian (3).

Notice that the integrals $F_1, \ldots, F_{n-1}$ have to be independent of x_n.

Theorem 1. *Let the Hamiltonian system with Hamiltonian* (2) *admit a full collection of the first integrals*

$$\{\mathcal{H} = F_1, F_2, \ldots, F_n\}$$

(that is, it is functionally independent and in involution), where the integrals $F_2, \ldots, F_n$ are polynomials in the momenta $p_1, p_2, \ldots, p_n$. Then this system also admits a full collection of integrals $\{\mathcal{H}_0, R_1, \ldots, R_{n-1}, p_n\}$, where the integrals R_i are polynomials in the momenta $p_1, p_2, \ldots, p_n$, independent of x_n.

Proof. Under conditions of the Theorem, rank $(\mathcal{H} = F_1, \dots, F_n) = n$ in all points of the phase space. Obviously, the elements of the collection Ω can be described in the form

$$F_i = F_{im_i} x_n^{m_i} + F_{im_i-1} x_n^{m_i-1} + \cdots + F_{i1} x_n + F_{i0}, \tag{4}$$

i.e. F_i are polynomials of degree m_i in the coordinate x_n, in addition $0 = m_1 \le m_2 \le \cdots \le m_n$. Denote

$$F_i^{(s)} = \frac{(m_i - s)!}{m_i!} \left(\frac{\partial}{\partial x_n} \right)^{(m_i - s)} F_i.$$

Then, evidently, $F_i^{(0)}$ is the coefficient F_{im_i} in the expansion (4).

Definition 1. The collection of the first integrals

$$\widetilde{\Omega} = \{\mathcal{H} = \widetilde{F}_1, \widetilde{F}_2, \dots, \widetilde{F}_n\}$$

is called to be a *minimal* collection of first integrals, if the following conditions hold:

1) the integrals $\widetilde{F}_i$ are put in ascending order of its degrees $\widetilde{m}_i$;
2) exactly k $(k > 0)$ of first integrals $\widetilde{F}_i$ are independent of x_n, i.e. $\widetilde{m}_k = 0 < \widetilde{m}_{k+1}$;
3) all the leading components $F_i^{(0)}$ $(k + 1 \le i \le n)$ are functionally independent with the collection $\{ \mathcal{H} = \widetilde{F}_1, \widetilde{F}_2, \dots, \widetilde{F}_k\}$;
4) all the leading components $F_i^{(0)}$ (including those for $1 \le i \le k$) are in involution;
5) Any collection $\widetilde{\widetilde{\Omega}}$, which satisfies the conditions 1–4, is not smaller (in lexicographic order with respect to the ascending sequence $\{m_i\}$) than the collection $\widetilde{\Omega}$.

Definition 2. The collection Ω of the first integrals is called to be *almost involutory*, if for any pair of the first integrals F_s and F_t, such that their orders m_s and m_t are nonzero, the following identity holds:

$$\{ (\partial/\partial x_n)^{m_s-1} F_s, (\partial/\partial x_n)^{m_t} F_t \} = \{ (\partial/\partial x_n)^{m_t-1} F_t, (\partial/\partial x_n)^{m_s} F_s \}.$$

It is clear that the class of the collections $\widetilde{\Omega}$, which satisfy the Definitions 1, 2, is nonempty. Therefore further the collection Ω will be considered to satisfy the Definitions 1, 2. Note that according to the Definition 2, we remove the restriction for all the first integrals F_i to be in involution.

Lemma 3. *If the collection Ω of n functionally independent first integrals is minimal and almost involutory, then $k > n - 3$.*

Proof. Assume that there is more than 2 first integrals, dependent on x_n, belonging to the collection Ω. It easy to show that the degrees of these integrals are equal to 1. Indeed, consider the collection $\omega = \{F_i \in \Omega \,|\, \deg_{x_n} F_i > 0\}$, i.e. $\omega = \{F_{k+1}, \dots, F_n\}$. Consider now the following two cases:

Case 1. $m_{k+1} = 1$. If $\deg_{x_n} F_n > 1$, then, according to the definition of minimal collection, $(\partial/\partial x_n)F_n$ is functionally dependent with $\{F_1, F_2, \dots, F_{n-1}\}$. At the same time, the collection $\widetilde{\omega} = \{F_{k+1}, \dots, F_n + F_{k+1}\}$ is also minimal, and for this modified set we also can state, that $(\partial/\partial x_n)(F_n + F_{k+1}$ is functionally dependent with $\{F_1, F_2, \dots, F_{n-1}\}$. It follows from two previous statements, that $(\partial/\partial x_n)F_{k+1} = F_{k+1}^{(0)}$ is functionally dependent with $\{F_1, F_2, \dots, F_{n-1}\}$. As $F_{k+1}^{(0)}$ does not depend on x_n, is in involution with the first integrals from the set $(\Omega\backslash\omega)$, then it