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Sets of Conditional Symmetry Operators and Exact Solutions for Wave and Generalised Heat Equations

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Reduction of some wave and heat equations by means of ansatzes in special coordinate systems results in PDEs with additional symmetry operators compared to original ones. Such operators will allow constructing sets of conditional symmetry operators that represent hidden symmetry of the equations. Classes of generalised wave and heat equations possessing these symmetry properties are described, and families of respective nonclassical solutions are listed.

1 Example 1: Linear wave equation

Let us consider the linear wave equation

$$\Box u = ku \tag{1}$$

for the real-valued function $u = u(x_0, x_1, x_2, x_3)$, $x_0 = t$ is the time variable, x_1, x_2, x_3 are space variables. $\Box$ is the d'Alembert operator

$$\Box u = -\frac{\partial^2 u}{\partial x_0^2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}.$$

The maximal invariance algebra (without trivial operators $u\partial_u$, $f(x)\partial_u$ ($\Box f = 0$) taken into account) of the equation (1), $k \neq 0$, is the Poincaré algebra $AP(1,3)$ with the basis operators

$$p_\mu = i g_{\mu\nu} \frac{\partial}{\partial x_\nu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu,$$

where μ, ν take the values 0, 1, 2, 3; $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$.

Similarity solutions for the equation (1) can be found by means of symmetry reduction by non-equivalent subalgebras of the algebra $AP(1,3)$ [1–4]. In such a way it is possible to obtain solutions in the form $u = \phi(\omega)$, where ω is an invariant of some subalgebra of the Poincaré algebra $AP(1,3)$.

Here we will look for solutions of the equation (1) in the form $u = f(x)\phi(\omega)$, with f and ω being some functions on x. Ansatzes of such form are usually obtained in the process of reduction of wave equations having scale invariance under subalgebras containing scale transformations $D = x^\nu p_\nu + \lambda i u \partial_u$. However, we can obtain a solution with such ansatzes for the equation (1) that is not invariant scale transformations.

We can reduce the equation (1) by means of the ansatz

$$u = f(r)\phi(\omega), \qquad r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \qquad \omega = r + v x_0.$$

The general form of the reduced equation will be

$$\left(v^2 - 1\right) f \phi'' - 2\left(f' + \frac{1}{r} f\right)\phi' - \left(f'' + \frac{2}{r} f' + kf\right)\phi = 0, \tag{2}$$

and the reduction conditions (when $f \neq 0$ and $\phi \neq 0$) are as follows: when v is an arbitrary parameter, $v^2 - 1 \neq 0$, then

$$f' + \frac{1}{r} f = \lambda f, \qquad f'' + \frac{2}{r} f' = \gamma f,$$

with λ, γ being arbitrary constants. This system is compatible if and only if $\gamma = \lambda^2$, and the ansatz takes the form

$$u = \frac{c \exp \lambda r}{r} \phi(\omega), \tag{3}$$

where $\omega = r + v x_0$; c, λ, $v \neq \pm 1$, $v \neq 0$ are arbitrary constants. The reduced equation has the form $\left(v^2 - 1\right) \phi'' - 2\lambda \phi' - \left(\lambda^2 + k\right) \phi = 0$.

When $v = \pm 1$, there may be several types of the reduction conditions.

1. With

$$f' + \frac{1}{r} f \neq 0, \qquad f'' + \frac{2}{r} f' + kf \neq 0, \qquad \alpha \neq 0,$$

the reduction condition will have the form

$$\alpha \left(f' + \frac{1}{r} f \right) = f'' + \frac{2}{r} f' + kf,$$

and the corresponding ansatzes are

$$u = \frac{1}{r} \exp \frac{\alpha r}{2} [c_1 \cosh \beta r + c_2 \sinh \beta r] \phi(\omega), \qquad 0 > k - \frac{\alpha^2}{4} = -\beta^2; \tag{4}$$

$$u = \frac{1}{r} \exp \frac{\alpha r}{2} [c_1 \cos \beta r + c_2 \sin \beta r] \phi(\omega), \qquad 0 < k - \frac{\alpha^2}{4} = \beta^2; \tag{5}$$

$$u = \frac{1}{r} \exp \frac{\alpha r}{2} [c_1 r + c_2] \phi(\omega), \qquad 0 = k - \frac{\alpha^2}{4}; \tag{6}$$

the reduced equation for ϕ will be $\phi' + \alpha \phi = 0$ and its solution $\phi = c \exp\left(-\frac{\alpha \omega}{2}\right)$ gives exact solutions for the equation (1).

2. With

$$f' + \frac{1}{r} f = 0, \qquad f'' + \frac{2}{r} f' + kf = 0,$$

ϕ may be an arbitrary function. It is evident that $f = 0$ if $k \neq 0$. If $k = 0$, $f = \frac{c}{r}$, and we obtain the well-known Euler solution for the wave equation $\Box u = 0$:

$$u = \frac{c}{r} \phi(r \pm t),$$

where ϕ is an arbitrary function. This solution is obtained from the degenerate case of the reduced equation (2) with all coefficients at and functions vanishing, so the solution of the resulting reduced equation is an arbitrary function. This solution corresponds to the hidden symmetry of the wave equation, represented by Lie infinite-dimensional symmetry algebra of the reduced equation $\phi_{tt} - \phi_{rr} = 0$, $u = \frac{c}{r} \phi(r, t)$.

3. With

$$f' + \frac{1}{r} f \neq 0, \qquad f'' + \frac{2}{r} f' + kf = 0,$$

$\phi' = 0$, $\phi = \text{const}$. The resulting solutions have the form $u = u(r)$ and are similarity solutions that may be obtained from the Lie symmetry of the equation (1).

It is quite obvious that ansatzes of the form (3)–(6) cannot be obtained by reduction of the equation (1) by subalgebras of the Poincaré algebra $AP(1,3)$

The ansatz (3) corresponds to the set of operators Q_1, J_{ab}, where $a,b=1,2,3$ and Q_1 has the form

$$Q_1 = x_a\partial_a - \frac{r}{v}\partial_0 + (\lambda r - 1)u\partial_u. \tag{7}$$

This set of operators determines conditional invariance of the equation (1), and the ansatz (3) allows to obtain new solutions compared to those that may be obtained by means of Lie symmetry operators. Note that these operators do not form an algebra.

We present other such operators $Q_{(1,i)}$ determining conditional invariance of the equation (1):

$$Q_{(1,1)} = x_a\partial_a + r\partial_0 + k^{1/2}ru\partial_u;$$
$$Q_{(1,2)} = x_a\partial_a + r\partial_0 + \left(\frac{\alpha}{2} - \frac{1}{r} + \left(k - \frac{\alpha^2}{4}\right)^{1/2}\operatorname{tg} r\right)u\partial_u;$$
$$Q_{(1,3)} = x_a\partial_a + r\partial_0 + \left(\frac{\alpha}{2} - \frac{1}{r} + \left(k - \frac{\alpha^2}{4}\right)^{1/2}\operatorname{ctg} r\right)u\partial_u.$$

These operators generate scale-type transformations of the space variables.

We adduce here some of the families of exact solutions for the equation (1):

$$u = \frac{c\exp\lambda r}{r}[c_1\exp(\lambda+\alpha)l\omega + c_2\exp(\lambda-\alpha)l\omega], \qquad 0 < v^2(k+\lambda^2) - k = \alpha^2;$$
$$u = \frac{c\exp(\lambda r + \lambda l\omega)}{r}[c_1\cos l\alpha\omega + c_2\sin l\alpha\omega], \qquad 0 > v^2(k+\lambda^2) - k = -\alpha^2;$$
$$u = \frac{c\exp(\lambda r + \lambda l\omega)}{r}[c_1\omega + c_2], \qquad v^2(k+\lambda^2) - k = 0,$$

$l = 1/(v^2-1)$; c_1, c_2 are arbitrary constants.

2 Example 2: Generalised heat equation

We consider a nonlinear generalised heat equation in the form

$$u_0 + u_{11} + u_{22} + u_{33} = F(x_0, x_1, x_2, x_3, u), \tag{8}$$

where $u_\mu = \frac{\partial u}{\partial x_\mu}$, $u_{\mu\mu} = \frac{\partial^2 u}{\partial x_\mu^2}$; $\mu = 0,1,2,3$.

Classical Lie symmetries with two and three spatial variables with F not depending on x_μ were described in [5].

We consider reductions of the equation (8) to an ODE by means of an ansatz of the form

$$u = \frac{c}{r}\phi(\omega), \qquad r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \qquad \omega = r + vx_0, \tag{9}$$

where v is an arbitrary parameter, when $F = \frac{1}{r}\Phi(\omega, ur)$. The reduced equation will then have the form $\phi'' + v\phi' = \Phi(\omega,\phi)$.

The maximal invariance algebra of the equation (8) for the general form of $F = \frac{1}{r}\Phi(\omega, ur)$ is the three-dimensional algebra of rotation operators with a basis J_{ab}, $a,b = 1,2,3$, and the ansatz (9) corresponds to the set of operators $\{Q_2, J_{ab}\}$, where $a,b=1,2,3$ and Q_2 has the form

$$Q_2 = x_a\partial_a - \frac{r}{v}\partial_0 + u\partial_u. \tag{10}$$

This set of operators determines conditional invariance of the equation (8), and the ansatz (9) allows to obtain new solutions compared to those that may be obtained by means of Lie symmetry operators. We will discuss the reduced equation for (8) by means of the ansatz (9) with $F = \lambda r^{k-1} u^k$ that has the form

$$\phi'' + v\phi' = \lambda\phi^k.$$

This equation may be transformed to the second-type Abel equation

$$p'p + vp = \lambda y^k \tag{11}$$

with substitution $\phi' = p$, $\phi = y$, $p = p(y)$.

1. If $k = \frac{1}{2}$ then the equation (11) by means of the transformation $\frac{s'}{s} = y^{1/2}$, $-\frac{1}{v}p = -\frac{z}{2A}$, $A = \frac{\lambda}{v^2}$ reduces to the Bessel equation

$$s'' - \frac{z}{2A}s = 0.$$

2. If $k = -1$ then the equation (11) is the Liouville equation $p'p + vp = \lambda y^{-1}$. By means of the substitution $-\frac{1}{v}p = y + z$, $y = \frac{1}{u}$ it is reduced to a linear one with respect to the function $u = u(z)$

$$\frac{\lambda}{v^2}u_z + zu + 1 = 0.$$

3. If $k = -2$ then the equation (11) by means of the substitution $-\frac{1}{v}p = y + z$, $\frac{1}{y} = u - \frac{z^2}{2}$ may be transformed to the Riccati equation for the function $z = z(u)$

$$\frac{\lambda}{v^2}z_u + u - \frac{z^2}{2} = 0,$$

whose solution may be expressed via the Bessel function.

3 Definitions and discussion

The concept of non-classical, or conditional symmetry, originated in its various facets in the papers [6–11], and later it was developed by numerous authors into the theory and a number of algorithms for studying symmetry properties of equations of mathematical physics and for construction of their exact solutions (see e.g. [12]). Here we will use the following definition of the conditional symmetry:

Definition 1. The equation $\Phi(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = 0$, where $\underset{k}{u}$ is the set of all kth-order partial derivatives of the function $u = (u^1, u^2, \ldots, u^m)$, is called conditionally invariant [3] under the operator

$$Q = \xi^i(x,u)\partial_{x_i} + \eta^r(x,u)\partial_{u^r}$$

if there is an additional condition

$$G(x, u, \underset{1}{u}, \ldots, \underset{l_1}{u}) = 0, \tag{12}$$

such that the system of two equations $\Phi = 0$, $G = O$ is invariant (according to the Lie definition, see e.g. [13]) under the operator Q.

If (12) has the form $G = Q[u]$, $Q[u]$ designates $Q[u] = \eta^r(x,u) - \xi^i(x,u)u_{x_i}$, then the equation $\Phi = 0$ is called Q-conditionally invariant under the operator Q [3]. These definitions of the conditional invariance of some equation are based on what is in reality Lie symmetry (see e.g. the classical texts [13–15]) of the same equation with a certain additional condition.

In this paper we develop results previously obtained by the authors in [16–18]. Our examples may be used to illustrate a generalisation of this definition to conditional invariance with a set of conditions $G_m = 0$. We may speak about $\langle Q_m\rangle$-conditional invariance where additional conditions have the form

$$G_m(x, u, \underset{1}{u}, \ldots, \underset{l_1}{u}) = Q_m[u] = 0.$$

However, we see that these examples contain features of two different types of non-classical invariance – hidden symmetry (see [30]) and Q-conditional symmetry.

Definition 2. An equation is said to have hidden conditional invariance if a reduced equation is conditionally invariant under some additional condition.

The equation (1) possesses hidden Q-conditional symmetry with respect to the operators Q_1, $Q_{1,i}$, and the equation (8) possesses hidden Q-conditional symmetry with respect to the operator Q_2, under conditions $J_{ab}u = 0$, $a, b = 1, 2, 3$.

The conditional symmetry of the Klein–Gordon equation was studied in [19, 20]. The conditional symmetry of the nonlinear heat equations with one space variable was extensively studied by numerous authors [6, 21–27, 29]. In [28] all conditional symmetries for an n-dimensional homogeneous linear heat equation were described.

The operators of the hidden conditional symmetry presented in this paper do not belong to sets of operators described in the above-mentioned papers, and they are not obtainable from conditional symmetry of equations with one space dimension. However, the operators we found are linked to the conditional symmetry of the respective radial wave and heat equations

$$u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F_1(u),$$
$$u_t + u_{rr} + \frac{n-1}{r}u_r = F_2(u)$$

(n is the number of spatial variables, here $n = 3$) that present considerable interest by themselves. See e.g. [31, 32], where exact solutions for a n-dimensional radial wave equation with a general power nonlinearity are derived and studied, and [33] for discussion of the linear radial heat equation.

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