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Symmetry Analysis of $(2 + 1)$-Dimensional Doebner–Goldin Equations

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The paper discusses the application of *MathLie* in connection with Lie group analysis. The examined example is the $(2 + 1)$-dimensional case of the Doebner–Goldin equations after Madelung representation. The related Lie algebras are calculated and classified. Furthermore we discuss the determination of an optimal system for the 7-dimensional case of one Lie algebra.

1 Introduction

The application of Lie's theory to examine systems of partial differential equations is one of the most efficient methods to calculate solutions for equations of motions. Furthermore Lie's theory allows the classification of solutions and related algebras. One can use Lie's transformation theory as a microscope to get information about the properties of a physical model [1, 2]. Lie's method is discussed in literature in great detail [3–6]. With the algorithms at hand one can generate computer programs such as *MathLie* to automatically carry out the calculations. Today there are a large number of symbolic computing programs for the algebraic manipulation of equations.

In this discussion we show investigations to get information about the structure of solutions of the Doebner–Goldin–Madelung equations. The symmetry investigations of these equations are carried out by using the Mathematica program *MathLie*. Section 3 is concerned with the algebra investigation of a 7-dimensional algebra using the *Mathematica* program *MathLieAlg* by R. Schmid [7]. Section 4 deals with an algorithm of calculating optimal systems. The example discussed in Section 4 is a 7-dimensional algebra generated by one of the Doebner–Goldin–Madelung equation.

2 Derivation of the Doebner–Goldin equations

The investigation of Borel quantization for S^1 leads to a non-linear Schrödinger equation of the form (here with $m = 1$, $\hbar = 1$):

$$\mathrm{i}\partial_t\psi = -\frac{1}{2}\Delta\psi + V(\vec{x},t)\psi + \frac{\mathrm{i}}{2}KR_2(\psi)\psi + \sum_{j=1}^{5} D_j R_j(\psi)\psi, \tag{1}$$

derived by Dobrev, Doebner and Twarock [8] and nowadays called Doebner–Goldin equations. Here, the $R_j(\psi)$ with $j \in \{1,\ldots,5\}$ are real valued functionals of the real valued density $\varrho = \overline{\varrho} = \psi\overline{\psi}$ and the real valued current $\vec{j} = \overline{\vec{j}} = \frac{\mathrm{i}\hbar}{2m}\left(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi\right)$ and can be found in [8, 9].

In the following we consider the $(2+1)$-dimensional case of equation (1) without potential V. Applying the Madelung transformation

$$\psi \to \sqrt{\varrho(\vec{x},t)}\exp\left(\mathrm{i}S(\vec{x},t)\right), \qquad \overline{\psi} \to \sqrt{\varrho(\vec{x},t)}\exp\left(-\mathrm{i}S(\vec{x},t)\right)$$

to equation (1) and dividing the resulting equations into real and imaginary parts we find the following system:

$$\varrho_t + S_x\varrho_x + S_y\varrho_y + \varrho S_{xx} + \varrho S_{yy} - \delta\varrho_{xx} - \delta\varrho_{yy} = 0, \tag{2}$$

$$\begin{aligned}(1+8D_5)\big(\varrho_x^2+\varrho_y^2\big) + 4\varrho^2\big(2S_t + S_x^2 + 2D_3S_x^2 + (1+2D_3)S_y^2 + 2D_1S_{xx} + 2D_1S_{yy}\big)\\ + 2\varrho(4D_1S_x\varrho_x + 4D_4S_x\varrho_x + 4(D_1+D_4)S_y\varrho_y - \varrho_{xx} + 4D_2\varrho_{xx} + (-1+4D_2)\varrho_{yy}) = 0,\end{aligned} \tag{3}$$

where $m = 1$ and $\hbar = 1$. Here, D_1, D_2, D_3, D_4, and δ are real valued parameters. By permutating these parameters we receive 63 different model equations (see table in [9]) of nonlinear Schrödinger type. This set of equations is called the set of Doebner–Goldin–Madelung equations. The whole set of equations are examined with the Mathematica program *MathLie*.

3 Symmetry analysis of the (2 + 1)-dimensional Doebner–Goldin–Madelung equations

In order to find the symmetry group of equations (2), (3), we apply the algorithms described in text books such as [3, 1]. We look for an algebra of vector fields of the form

$$V = \xi[1]\partial_x + \xi[2]\partial_t + \phi[1]\partial_\varrho + \phi[2]\partial_S,$$

where the infinitesimals $\xi[1]$, $\xi[2]$ and $\phi[1]$, $\phi[2]$ depend on x, t, ϱ, and S in general.

These coefficients are determined from the requirement that the second prolongation of V should annihilate the equation on the solution set of the equation. This was done by using the Mathematica program *MathLie* [3] for all 63 model equations of the Doebner–Goldin–Madelung equations.

The next step of our discussion is related to the investigation of the 7-dimensional algebra of the equation with the parameters D_3, D_5 (see [9]). We use a *Mathematica* program *MathLieAlg* by R. Schmid [7]. The generators of this equation are:

$$\begin{aligned}&V[1] = \partial_t, \quad V[2] = \varrho\partial_\varrho, \quad V[3] = -4t\partial_t - 2x\partial_x - 2y\partial_y,\\ &V[4] = \partial_x, \quad V[5] = \partial_S, \quad V[6] = -y\partial_x + x\partial_y, \quad V[7] = \partial_y.\end{aligned} \tag{4}$$

The only non-zero commutators of the vector fields are following:

$$\begin{aligned}&[V[1], V[3]] = -4V[1], \quad [V[3], V[4]] = 2V[4], \quad [V[3], V[7]] = 2V[7],\\ &[V[4], V[6]] = V[7], \quad [V[6], V[7]] = V[4].\end{aligned}$$

Nontrivial algebra elements are $\{V[1], V[2], V[3], V[5], V[6]\}$. Algebras can be generated by the following sets:

$$\begin{aligned}&\{V[1], V[2], V[3], V[4], V[5], V[6]\}, \quad \{V[1], V[2], V[3], V[5], V[6], V[7]\},\\ &\{V[1], V[2], V[3], V[4], V[5], V[6], V[7]\}.\end{aligned}$$

The Cartan metric of this algebra reads:

$$\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&24&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-2&0\\ 0&0&0&0&0&0&0 \end{pmatrix}.$$

The algebraic properties are following: not semisimple, solvable, not nilpotent.

From this algebra the subalgebras listed in Table 1 can be derived:

Table 1. Table of subalgebras[1].

Dimension	Number of subalgebras	Subalgebra elements
0	1	$\{\}$
1	7	$\{V[1]\},\{V[2]\},\{V[3]\},\{V[4]\},\{V[5]\},\{V[6]\},\{V[7]\}$
2	19	$\{V[1],V[2]\},\{V[1],V[3]\},\{V[1],V[4]\},\{V[1],V[5]\},\{V[1],V[6]\},$ $\{V[1],V[7]\},\{V[2],V[3]\},\{V[2],V[4]\},\{V[2],V[5]\},\{V[2],V[6]\},$ $\{V[2],V[7]\},\{V[3],V[4]\},\{V[3],V[5]\},\{V[3],V[6]\},\{V[3],V[7]\},$ $\{V[4],V[5]\},\{V[4],V[7]\},\{V[5],V[6]\},\{V[5],V[7]\}$
3	27	$\{V[1],V[2],V[3]\},\{V[1],V[2],V[4]\},\{V[1],V[2],V[5]\},\{V[1],V[2],V[6]\},$ $\{V[1],V[2],V[7]\},\{V[1],V[3],V[4]\},\{V[1],V[3],V[5]\},\{V[1],V[3],V[6]\},$ $\{V[1],V[3],V[7]\},\{V[1],V[4],V[5]\},\{V[1],V[4],V[7]\},\{V[1],V[5],V[6]\},$ $\{V[1],V[5],V[7]\},\{V[2],V[3],V[4]\},\{V[2],V[3],V[5]\},\{V[2],V[3],V[6]\},$ $\{V[2],V[3],V[7]\},\{V[2],V[4],V[5]\},\{V[2],V[4],V[7]\},\{V[2],V[5],V[6]\},$ $\{V[2],V[5],V[7]\},\{V[3],V[4],V[5]\},\{V[3],V[4],V[7]\},\{V[3],V[5],V[6]\},$ $\{V[3],V[5],V[7]\},\{V[4],V[5],V[7]\},\{V[4],V[6],V[7]\}$
4	23	$\{V[1],V[2],V[3],V[4]\},\{V[1],V[2],V[3],V[5]\},\{V[1],V[2],V[3],V[6]\},$ $\{V[1],V[2],V[3],V[7]\},\{V[1],V[2],V[4],V[5]\},\{V[1],V[2],V[4],V[7]\},$ $\{V[1],V[2],V[5],V[6]\},\{V[1],V[2],V[5],V[7]\},\{V[1],V[3],V[4],V[5]\},$ $\{V[1],V[3],V[4],V[7]\},\{V[1],V[3],V[5],V[6]\},\{V[1],V[3],V[5],V[7]\},$ $\{V[1],V[4],V[5],V[7]\},\{V[1],V[4],V[6],V[7]\},\{V[2],V[3],V[4],V[5]\},$ $\{V[2],V[3],V[4],V[7]\},\{V[2],V[3],V[5],V[6]\},\{V[2],V[3],V[5],V[7]\},$ $\{V[2],V[4],V[5],V[7]\},\{V[2],V[4],V[6],V[7]\},\{V[3],V[4],V[5],V[7]\},$ $\{V[3],V[4],V[6],V[7]\},\{V[4],V[5],V[6],V[7]\}$
5	13	$\{V[1],V[2],V[3],V[4],V[5]\},\{V[1],V[2],V[3],V[4],V[7]\},\{V[1],V[2],V[3],V[5],V[6]\},$ $\{V[1],V[2],V[3],V[5],V[7]\},\{V[1],V[2],V[4],V[5],V[7]\},\{V[1],V[2],V[4],V[6],V[7]\},$ $\{V[1],V[2],V[3],V[4],V[5]\},\{V[1],V[3],V[4],V[5],V[7]\},\{V[1],V[3],V[4],V[6],V[7]\},$ $\{V[1],V[4],V[5],V[6],V[7]\},\{V[2],V[3],V[4],V[5],V[7]\},\{V[2],V[3],V[4],V[6],V[7]\},$ $\{V[2],V[4],V[5],V[6],V[7]\},\{V[3],V[4],V[5],V[6],V[7]\}$
6	5	$\{V[1],V[2],V[3],V[4],V[5],V[7]\},\{V[1],V[2],V[3],V[4],V[6],V[7]\},$ $\{V[1],V[2],V[4],V[5],V[6],V[7]\},\{V[1],V[3],V[4],V[5],V[6],V[7]\},$ $\{V[2],V[3],V[4],V[5],V[6],V[7]\}$
7	1	$\{V[1],V[2],V[3],V[4],V[5],V[6],V[7]\}$

Examining the subalgebras we can find the ideals listed in [9]. The normalizer of all ideals is the algebra $\{V[1], V[2], V[3], V[4], V[5], V[6], V7]\}$ and the radical reads $\{V[1], V[2], V[3], V[4], V[5], V[6], V7]\}$. For the center we find $\{V[2], V[5]\}$. The adjoint representation of our 7-dimensional algebra can be found in [9].

4 The optimal system of the seven-dimensional algebra

Let us first consider the general system of differential equations

$$F\big(x, u, u^{(n)}\big) = 0 \tag{5}$$

and the related symmetry group G. For every s-parametric subgroup H_s one can calculate similarity solutions under the assumption[2], that $s < \min(r, n')$, where n' is the number of independent variables and r is the order of the system of differential equations [4]. In this set of similarity solutions there are such solutions, which can be calculated by a transformation of the symmetry group from other similarity solutions. Our aim is to derive a minimal set of similarity solutions from which one can gain all the other solutions by a transformation. Such a list is called optimal system of similarity solutions with elements which are essentially different types of similarity solutions. The application of the conjugation (see e.g. [12]) and a theorem

[1] $\{\}$ is empty set.

[2] Following [12] the number of parameters s of a subgroup H of a symmetry group G will be written under the symbol of the subgroup.

by Olver [6] allows us to transform the problem of classifying solutions to that of classifying subgroups. The adjoint representation maps this problem to the classification of subalgebras with respect to inner automorphisms. The result is the optimal system of subalgebras. More details can be found in [4, 6].

The literature presents several methods of classifying subalgebras. A detailed discussion of the procedures are given in [10, 4, 5, 11]. A common property of these methods is that they all start with algebras of very low dimension.

Due to Ovsyannikov [13–15] we organize our calculations by the following definition and theorem.

Definition 1 ([15]). An optimal system ΘL is said to be normalized if $\operatorname{Nor} K \in \Theta L$ whenever a subalgebra K is in ΘL.

The existence of such optimal system follows from

Theorem 1 ([15]). *For any finite-dimensional Lie algebra there exists a normalized optimal system ΘL of subalgebras.*

The following discussion demonstrates the application of our algorithm [2] to the 7-dimensional algebra 4. We start our calculation with the series of ideals

$$\begin{aligned}&\{\} \subset \{V[1], V[4]\} \subset \{V[1], V[4], V[7]\} \subset \{V[1], V[3], V[4], V[5], V[6], V[7]\}\\ &\quad \subset \{V[1], V[2], V[3], V[4], V[5], V[6], V[7]\} = L_7\end{aligned}$$

and take the maximal Abelian ideal $I_{\max} = \{V[1], V[4], V[7]\}$. The related factor algebra $L_7 \backslash I_{\max}$ is $\{V[2], V[3], V[5], V[6]\}$. Now we are doing the same step with this factor algebra. The series of ideals is

$$0 \subset \{V[2], V[5]\} \subset \{V[2], V[3], V[5], V[6]\}$$

with the maximal Abelian subalgebra $\{V[2], V[5]\}$. The related factor algebra reads $\{V[3], V[6]\}$. By starting with the smallest factoralgebra we have to express a general vector Y by a linear combination of the vectors $V[3]$, $V[6]$:

$$Y_1 = x_3 V[3] + x_6 V[6], \qquad Y_2 = y_3 V[3] + y_6 V[6]. \tag{6}$$

The coefficient matrix of (6) reads $\begin{pmatrix} x_3 & x_6 \\ y_3 & y_6 \end{pmatrix}$ which allows manipulation by row and application of the adjoint representation of the ranks 2, 1, 0. The classification of the above matrix delivers the result in Table 2.

Table 2. Optimal system of the two-dimensional subalgebra $\{V[3], V[6]\}$.

Dimension	Subalgebras
2	$\{V[3], V[6]\}$
1	$\{V[3] + x_6 V[6]\}$, $x_6 \neq 0$, $\{V[3]\}$, $\{V[6]\}$
0	$\{\}$

Now we have to consider the first extension of the algebra. An equation similar to (6) leads to the matrix

$$\begin{pmatrix} u_2 & u_5 & u_3 & u_6 \\ v_2 & v_5 & v_3 & v_6 \\ x_2 & x_5 & x_3 & x_6 \\ y_2 & y_5 & y_3 & y_6 \end{pmatrix} \tag{7}$$

which has to be classified. It allows the ranks 4, 3, 2, 1, 0. In addition there exists a block structure. The 2×2 matrix in the lower right corner with the indices $(3,6)$ and the matrix in the upper left corner with the indices $(2,5)$.

We start our calculation with the matrix in the upper left corner related to the indices $(2,5)$, which allows the ranks 2, 1, 0. The results are

Table 3. Optimal system of the subalgebra $\{V[2], V[5]\}$.

Dimension	Subalgebras
2	$\{V[2], V[5]\}$
1	$\{V[2] + u_5 V[5]\}$, $u_5 \neq 0$, $\{V[2]\}$, $\{V[5]\}$

For the rank 0 the upper left matrix $(2,5)$ only contains 0. So we have to consider the matrix containing the last two lines of (7). The block matrix with $(3,6)$ index runs through all subalgebras of Table 2. The final result of this calculation step is

Table 4. Optimal system of the subalgebra $\{V[2], V[5], V[3], V[6]\}$.

Dimension	Subalgebras
4	$\{V[3], V[6], V[2], V[5]\}$
3	$\{V[3], V[6], V[2] + x5V[5]\}, \{V[3], V[6], V[2]\}, \{V[3], V[6], V[5]\}$ $\{V[3] + x6V[6], V[2], V[5]\}, \{V[3], V[2], V[5]\}, \{V[6], V[2], V[5]\}$
2	$\{V[3] + x_6V[6], V[2] + x_5V[5]\}, \{V[3] + x_6V[6], V[2]\}, \{V[3] + x_6V[6], V[5]\}$, $\{V[3], V[2] + x_5V[5]\}, \{V[3], V[2]\}, \{V[3], V[5]\}, \{V[6], V[2] + x_5V[5]\}, \{V[6], V[2]\}$, $\{V[6], V[5]\}, \{V[2], V[5]\}, \{x_2V[2] + x_5V[5] + V[3], y_2V[2] + y_5V[5] + V[6]\}$, $\{x_2V[2] + x_5V[5] + V[3] + x_6V[6], y_2V[2] + y_5V[5]\}, \{x_2V[2] + x_5V[5] + V[3], y_2V[2] + y_5V[5]\}$, $\{x_2V[2] + x_5V[5] + V[6], y_2V[2] + y_5V[5]\}, \{V[3], V[6]\}$
1	$\{V[2] + x_5V[5]\}, \{V[2]\}, \{V[5]\}, \{V[3] + x_6V[6]\}, \{V[3]\}, \{V[6]\}$

In the next step of our algorithm [2], we have to consider the properties of the matrix

$$\begin{pmatrix} a_1 & a_4 & a_7 & a_5 & a_3 & a_6 \\ b_1 & b_4 & b_7 & b_5 & b_3 & b_6 \\ c_1 & c_4 & c_7 & c_5 & c_3 & c_6 \\ u_1 & u_4 & u_7 & u_5 & u_3 & u_6 \\ v_1 & v_4 & v_7 & v_5 & v_3 & v_6 \\ x_1 & x_4 & x_7 & x_5 & x_3 & x_6 \\ y_1 & y_4 & y_7 & y_5 & y_3 & y_6 \end{pmatrix}. \tag{8}$$

It desintegrates into a block structure where we have to consider the matrix in the upper left corner (first three lines) with the indices $(1,3,7)$. It allows the ranks 3, 2, 1, 0. The result of this classification is

Table 5. Optimal system of the subalgebra $\{V[1], V[4], V[7]\}$.

Dimension	Subalgebras
3	$\{V[1], V[4], V[7]\}$
2	$\{V[1] + a_7V[7], V[4] + b_7V[7]\}, \{V[1] + a_4V[4], V[7]\}, \{V[4], V[7]\}$
1	$\{V[1] + a_4V[4] + a_7V[7]\}, \{V[1] + a_7V[7]\}, \{V[1]\}, \{V[4] + a_7V[7]\}, \{V[7]\}$

By taking into account the block structure we have the matrix in the upper left corner with the indices $(1,4,7)$ (first three lines) and the matrix in the right left corner with the indices $(2,5,3,6)$ (last four lines). To classify the whole matrix (8) every subalgebra of the upper left corner from Table 5 has to combine with every subalgebra of the lower right corner from Table 4.

For the rank 0 we have to investigate the matrix built by the last four lines of (8). The block matrix with the indices $(2,5,3,6)$ runs through all subalgebras of Table 4. The final result of classifying this matrix is given in Table 6.

Table 6. Optimal system of the subalgebra $\{V[1], V[4], V[7], V[2], V[5], V[3], V[6]\}$.

Dimension	Optimal system
4	$\{V[3], v_1V[1] + V[6], x_1V[1] + V[2] + x_5V[5], y_1V[1]\}, \{V[3], v_1V[1] + V[6], V[5], y_1V[1]\},$
	$\{V[2] + u_7V[7], V[5] + v_7V[7], V[3], y_7V[7]\}, \{V[2] + u_4V[4] + u_7V[7], V[5], V[3], y_4V[4] + y_7V[7]\},$
	$\{V[2] + u_4V[4], V[3] + v_6V[6], V[4], V[7]\}, \{V[2] + u_7V[7], V[3] + v_6V[6], V[4], V[7]\},$
	$\{V[5] + u_7V[7], V[3] + v_6V[6], V[4], V[7]\}, \{u_4V[4] + V[5], V[3] + v_6V[6], V[4], V[7]\},$
	$\{V[2] + u_5V[5] + u_7V[7], V[3], V[4], V[7]\}, \{V[2] + u_5V[5], V[3], V[1], V[4] + y_7V[7]\},$
	$\{u_1V[1] + V[2] + u_5V[5], V[3], V[1], V[7]\}, \{V[2] + u_4V[4] + u_5V[5], V[3], V[4], V[7]\},$
	$\{u_1V[1] + V[2] + u_7V[7], V[3], V[1], V[7]\}, \{u_1V[1] + V[2] + u_7V[7], V[3], V[4], V[7]\},$
	$\{u_1V[1] + V[2] + u_4V[4], V[3], V[1], V[4]\}, \{u_1V[1] + V[2], V[3], V[1], V[7]\},$
	$\{V[2] + u_4V[4], V[3], V[4], V[7]\}, \{V[5] + u7V[7], V[3], V[4], V[7]\},$
	$\{u_1V[1] + V[5], V[3], V[1], V[4] + y_7V[7]\}, \{u_1V[1] + V[5], V[3], V[1], V[7]\},$
	$\{u_4V[4] + V[5], V[3], V[4], V[7]\}, \{u_1V[1] + V[6], v_1V[1] + V[2] + v_5V[5] + v_7V[7], V[4], V[7]\},$
	$\{u_1V[1] + V[6], v_1V[1] + V[2] + v_4V[4] + v_5V[5], V[4], V[7]\},$
	$\{u_1V[1] + V[6], v_1V[1] + V[2] + v_7V[7], V[4], V[7]\}, \{u_1V[1]+V[6], v_1V[1]+V[2]+v_4V[4], V[4], V[7]\},$
	$\{u_1V[1] + V[6], v_1V[1] + V[5] + v_7V[7], V[4], V[7]\}$
	$\{u_1V[1] + V[6], v_1V[1] + v_4V[4] + V[5], V[4], V[7]\},$
	$\{u_1V[1] + V[2] + u_7V[7], v_1V[1] + v_4V[4] + V[5] + x_7V[7], V[1] + x_7V[7], V[4] + y_7V[7]\},$
	$\{u_1V[1] + V[2] + u_7V[7], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[1] + x_4V[4], V[7]\},$
	$\{u_1V[1] + V[2] + u_7V[7], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[4], V[7]\},$
	$\{u_1V[1] + V[2] + u_4V[4], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[1] + x_7V[7], V[4] + y_7V[7]\},$
	$\{u_1V[1] + V[2] + u_4V[4], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[1] + x_4V[4], V[7]\},$
	$\{u_1V[1] + V[2] + u_4V[4], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[4], V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5], v_2V[2] + v_4V[4] + V[6] + v_7V[7], V[4], V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5], v_2V[2] + v_4V[4] + V[6], V[4], V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5] + u_6V[6], v_2V[2] + v_5V[5] + v_7V[7], V[4], V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5] + u_6V[6], v_2V[2] + v_4V[4] + v_5V[5], V[4], V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5], v_1V[1] + v_2V[2] + v_5V[5] + v_7V[7], V[4], V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5], v_1V[1] + v_2V[2] + v_5V[5], V[1], V[4] + y_7V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5], v_1V[1] + v_2V[2] + v_5V[5], V[1], V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5], v_2V[2] + v_4V[4] + v_5V[5], V[4], V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5], v_2V[2] + v_5V[5] + V[6] + v_7V[7], V[4], V[7]\},$
	$\{u_2V[2] + V[3] + u_5V[5], v_1V[1] + v_2V[2] + v_4V[4] + v_5V[5] + V[6], V[1], V[4]\},$
	$\{u_2V[2] + V[3] + u_5V[5], v_2V[2] + v_4V[4] + v_5V[5] + V[6], V[4], V[7]\},$
	$\{u_1V[1] + u_2V[2] + u_5V[5] + V[6], v_1V[1] + v_2V[2] + v_5V[5] + v_7V[7], V[4], V[7]\},$
	$\{u_1V[1] + u_2V[2] + u_5V[5] + V[6], v_1V[1] + V[2] + v_4V[4] + v_5V[5], V[4], V[7]\},$
	$\{V[3], V[6] + v_7V[7], V[4], V[7]\}, \{V[3], v_4V[4] + V[6], V[4], V[7]\},$
	$\{u_1V[1] + V[2] + u_5V[5] + V[6], V[1], V[4], V[7]\},$
	$\{u_1V[1] + V[2] + u_4V[4] + v_5V[5], V[1], V[4], V[7]\},$
	$\{u_1V[1] + V[2] + u_4V[4] + v_5V[5], V[1], V[4], V[7]\},$
	$\{u_1V[1] + V[2] + v_7V[7], V[1], V[4], V[7]\}, \{u_1V[1] + V[2] + v_4V[4], V[1], V[4], V[7]\},$
	$\{u_1V[1] + V[5] + u_7V[7], V[1], V[4], V[7]\}, \{u_1V[1] + u_4V[4] + V[5], V[1], V[4], V[7]\},$
	$\{V[3] + u_6V[6], V[1], V[4], V[7]\}, \{V[3], V[1], V[4], V[7]\}, \{u_1V[1] + V[6], V[1], V[4], V[7]\}$
3	$\{V[3], V[6], V[2] + x_5V[5]\}, \{V[3], V[6], V[2]\}, \{V[3], V[6], V[5]\}, \{V[2], V[5], V[3] + x_6V[6]\},$
	$\{V[2], V[3], V[5]\},\{u_1V[1] + V[2], v_1V[1] + V[5], x_1V[1] + V[6]\}, \{u_1V[1] + V[2], V[3] + v_6V[6], V[1]\},$
	$\{u_1V[1] + V[5], V[3] + v_6V[6], V[1]\}, \{u_1V[1] + V[2] + u_5V[5], V[3], V[1]\},$
	$\{V[2] + u_5V[5], V[3], V[4] + x_7V[7]\}, \{V[2] + u_4V[4] + u_5V[5], V[3], V[4]\},$
	$\{V[2] + u_5V[5], V[3], V[7]\}, \{u_1V[1] + V[2], V[3], V[1]\}, \{V[2], V[3], V[4] + x_7V[7]\},$
	$\{V[2] + u_4V[4], V[3], V[4]\}, \{V[2], V[3], V[7]\}, \{V[5] + u_7V[7], V[3], V[7]\},$
	$\{u_1V[1] + V[5], V[3], V[1]\},$
	$\{V[5], V[3], V[4] + x_7V[7]\}, \{u_4V[4] + V[5], V[3], V[4]\}, \{V[5], V[3], V[7]\},$
	$\{u_1V[1] + V[6], v1V[1] + V[2] + v_5V[5], V[1]\}, \{u_1V[1] + V[6], v1V[1] + V[2], V[1]\},$
	$\{u_1V[1] + V[2] + u_7V[7], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[1] + x_4V[4] + x_7V[7]\},$
	$\{u_1V[1] + V[2] + u_7V[7], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[1] + x_7V[7]\},$
	$\{u_1V[1] + V[2] + u_7V[7], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[1]\},$
	$\{u_1V[1] + V[2] + u_7V[7], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[4] + x_7V[7]\},$
	$\{u_1V[1] + V[2] + u_7V[7], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[4]\},$
	$\{u_1V[1] + V[2] + u_7V[7], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[7]\},$
	$\{u_1V[1] + V[2] + u_4V[4], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[1] + x_4V[4] + x_7V[7]\},$
	$\{u_1V[1] + V[2] + u_4V[4], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[1] + x_7V[7]\},$
	$\{u_1V[1] + V[2] + u_4V[4], v_1V[1] + v_4V[4] + V[5] + v_7V[7], V[1]\},$

Dimension	Optimal system
3	$\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[4]\}$,
	$\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[7]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_1V[1]+v_2V[2]+v_5V[5]+V[6], V[1]\}$,
	$\{u_2V[2]+V[3]+u_5V[5]+u_6V[6], v_1V[1]+v_2V[2]+v_5V[5], V[1]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_1V[1]+v_2V[2]+v_5V[5], V[1]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5], V[4]+x_7V[7]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_4V[4]+v_5V[5], V[4]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5], V[7]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5]+V[6], V[1]\}$,
	$\{V[3], v_1V[1]+V[6], V[1]\}, \{u1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], v_4V[4], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[1]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[4], V[7]\}, \{u_1V[1]+V[2]+u_4V[4], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4], V[1]+v_4V[4], V[7]\}, \{u_1V[1]+V[2]+u_4V[4], V[4], V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[1]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[4], V[7]\}, \{u_1V[1]+u_4V[4]+V[5], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+u_4V[4]+V[5], V[1]+v_4V[4], V[7]\}, \{u_1V[1]+u_4V[4]+V[5], V[4], V[7]\}$,
	$\{V[3]+u_6V[6], V[4], V[7]\}, \{V[3], V[1]+v_7V[7], V[4]+x_7V[7]\}, \{V[3], V[1]+v_4V[4], V[7]\}$,
	$\{V[3], V[4], V[7]\}, \{u_1V[1]+V[6], V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5],+u_7V[7], V[1]+v_4V[4]+V[7]\}$
2	$\{V[2]+u_5V[5], V[3]+v_6V[6]\}, \{V[2], V[3]+v_6V[6]\}, \{u_1V[1]+V[5], V[3]+v_6V[6]\}$,
	$\{V[2]+u_5V[5], V[3]\}, \{V[2], V[3]\}, \{V[5], V[3]\}, \{u_1V[1]+V[6], v_1V[1]+V[2]+v_5V[5]\}$,
	$\{u_1V[1]+V[6], v_1V[1]+V[2]\}, \{u_1V[1]+V[6], v_1V[1]+V[5]\}$,
	$\{u_1V[1]+u_7V[7]+V[2], v_1V[1]+v_4V[4]+v_7V[7]+V[5]\}$,
	$\{u_1V[1]+u_4V[4]+V[2], v_1V[1]+v_4V[4]+v_7V[7]+V[5]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5]\}, \{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5]+V[6]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5]+V[6]\}$,
	$\{u_1V[1]+u_2V[2]+u_5V[5]+V[6], v_1V[1]+v_2V[2]+v_5V[5]\}$,
	$\{V[3], V[6]\}, \{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_4V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_7V[7]\}, \{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[4]+v_7V[7]\}, \{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[4]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]+v_4V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]+v_7V[7]\}, \{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[4]+v_7V[7]\}, \{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[4]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[7]\}, \{u_1V[1]+V[2]+u_7V[7], V[1]+v_4V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[1]+v_7V[7]\}, \{u_1V[1]+V[2]+u_7V[7], V[1]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[4]+v_7V[7]\}, \{u_1V[1]+V[2]+u_7V[7], V[4]\}$,
	$\{u_1V[1]+V[2]+v_4V[4], V[1]+v_4V[4]+v_7V[7]\}, \{u_1V[1]+V[2]+v_4V[4], V[1]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+v_4V[4], V[1]\}, \{u_1V[1]+V[2]+v_4V[4], V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+v_4V[4], V[4]\}, \{u_1V[1]+V[2]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[1]+v_4V[4]+v_7V[7]\}, \{u_1V[1]+V[5]+u_7V[7], V[1]+v_7V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[1]\}, \{u_1V[1]+V[5]+u_7V[7], V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[4]\}, \{u_1V[1]+V[5]+u_7V[7], V[7]\}$,
	$\{u_1V[1]+u_4V[4]+V[5], V[1]+v_4V[4]+v_7V[7]\}, \{u_1V[1]+u_4V[4]+V[5], V[1]+v_7V[7]\}$,
	$\{u_1V[1]+u_4V[4]+V[5], V[1]\}, \{u_1V[1]+u_4V[4]+V[5], V[4]+v_7V[7]\}$,
	$\{u_1V[1]+u_4V[4]+V[5], V[4]\}, \{u_1V[1]+u_4V[4]+V[5], V[7]\}, \{V[3]+u_6V[6], V[1]\}$,
	$\{V[3], V[1]+v_4V[4]+v_7V[7]\}, \{V[3], V[1]+v_7V[7]\}, \{V[3], V[1]\}, \{V[3], V[4]+v_7V[7]\}$,
	$\{V[3], V[4]\}, \{V[3], V[7]\}, \{u_1V[1]+V[6], V[1]\}$
1	$\{u_1V[1]+u_7V[7]+V[2]+u_5V[5]\}, \{u_1V[1]+u_4V[4]+V[2]+u_5V[5]\}$,
	$\{u_1V[1]+u_7V[7]+V[2]\}, \{u_1V[1]+u_4V[4]+V[2]\}$,
	$\{u_1V[1]+u_7V[7]+V[5]\}, \{u_1V[1]+u_4V[4]+V[5]\}, \{V[3]\}$

Therefore the subalgebras of L_7 are classified. We can find one 7-dimensional, 9 six-dimensional, 39 five-dimensional, 145 four-dimensional, 168 three-dimensional, 88 two-dimensional, 7 one-dimensional and one zero-dimensional subalgebras.

5 Conclusion

In our examination we have calculated the infinitesimals for each equation of the set of Doebner–Goldin–Madelung models. We found 10-, 8- and 7-dimensional algebras which were investigated. For the 7-dimensional algebra we have determined the optimal systems.

By application of the statements at the beginning of Section 4 we can calculate for every one and two-dimensional optimal-system the related reduction. In the case of the one-dimensional optimal system the result is a system of equations with two new independent variables whether for the 2-dimensional optimal system the reduced system will be an ordinary differential equation system which can be solved. More details can be found in [2]. In this situation we want to emphasize that it is necessary to take isomorphic investigations into account. A closer look at the commutator table shows that the algebras of each dimension look very similar.

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