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On Certain Quotient of Temperley–Lieb Algebra

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We consider a certain quotient of Temperley–Lieb algebra of general Coxeter system. This algebra arises as a quotient of Hecke algebra by some more relations than a Temperley–Lieb one. These additional relations correspond to the pairs of commuting generators of Coxeter system in the same way as Temperley–Lieb relations correspond to noncommuting generators. We prove that such algebras are usually well behaved. Especially for an irreducible system whose Coxeter graph does not contain a cycle this quotient is a sum of a matrix algebra and a field for all except a finite set of values of parameter.

1 Hecke deformation of Coxeter group algebra and its quotients

In this section we associate an algebra to a Coxeter group. This algebra is a certain quotient of the Temperley–Lieb algebra associated to the same Coxeter group. In further sections we show that such an algebra usually has simple enough structure.

Let W be a Coxeter group with set $S \subset W$ of generating reflections, let $l : W \longrightarrow \mathbb{N}_0$ be length correspondent to generators S. Let $q \in \mathbb{C}$.

Definition 1. The Hecke algebra $H_q = H_q(W)$ is an associative $\mathbb{C}$-algebra with $\mathbf{1}$, spanned by elements $\{T_w; w \in W\}$ with multiplication

$$T_s T_w = \begin{cases} T_{sw}, & l(sw) = l(w) + 1; \\ (q-1)T_w + qT_{sw}, & l(sw) = l(w) - 1, \end{cases}$$

for $s \in S$ and $w \in W$. Here $T_{\mathbf{1}} = \mathbf{1}_{H_q}$.

Note that H_q is a deformation of Coxeter group algebra $H_1 = \mathbb{C}W$, and its dimension (growth) is not greater then for $\mathbb{C}W$. For the general theory of Hecke algebras arising from Coxeter systems see [1, 2].

Consider for any pair of different reflections $s_i, s_j \in S$ a subgroup $W_{i,j} \subset W$ generated by s_i and s_j. $W_{i,j}$ is a dihedral group with $2m$ elements, where m is an order of element $s_i s_j$ in W.

Definition 2. The *complete* Temperley–Lieb algebra $TL_q^c = TL_q^c(W)$ is a quotient of $H_q(W)$ by relations

$$\sum_{w \in W_{i,j}} T_w = 0 \tag{1}$$

for each pair of different generators $s_i, s_j \in S$.

Note that quotient of $H_q(W)$ by relations (1) for each pair of noncommuting generators $s_i, s_j \in S$ is called the Temperley–Lieb algebra $TL_q = TL_q(W)$ (see [3] and references there). So, TL_q^c is a quotient of TL_q.

Remark 1. There is a way to consider Hecke and Temperley–Lieb algebras where q is indeterminate. For example, in [4, 5] this is done for Coxeter groups of type A (symmetric groups). That is well known that Temperley–Lieb algebra of symmetric group S_n over $\mathbb{C}(q)$ is semisimple, and its evaluation at $q \in \mathbb{C}\backslash\{0\}$ is also semisimple if and only if $q + q^{-1} + 2 \neq 4\cos^2 \frac{\pi}{m}$ for some $2 \leq m \leq n$ (see, for example, [2]). In [3] Hecke and Temperley–Lieb algebras of general Coxeter groups are considered over the ring of Laurent polynomials $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ where $q = v^2$. Then a set $\{T_w; w \in W\}$ would be an $\mathcal{A}$-basis of Hecke algebra. In [6] $\mathcal{A}$-basis of Temperley–Lieb algebra TL_q is given as a certain subset of $\{T_w; w \in W\}$.

In this paper all algebras will be algebras with **1** over field $\mathbb{C}$. In the following section we apply combinatorial argumentation based on Diamond lemma to algebras TL_q^c. A restriction to algebras over a field (not ring) is essential here. For Diamond lemma and Groebner basis see, for example, [7].

2 Presentation of TL_q^c via idempotents. Linear basis

Now (and till the end of the paper) we restrict to the class of Coxeter groups (W, S) where $S = \{s_1, \dots, s_n\}$ is a finite set and for each $1 \leq i < j \leq n$ an element $s_i s_j$ has an order either 2 or 3. Let also $q \neq -1, 0$.

In this section we propose another presentation of an algebra $TL_q^c(W)$, which is useful to find its linear basis and calculate the dimension (or Gelfand–Kirillov dimension).

Consider $p_i = \frac{T_{s_i}+1}{q+1} \in H_q$, and $\tau = \frac{q}{(1+q)^2}$. Then $p_i^2 = p_i$. Consider two cases.

If s_i and s_j commute in W, then T_{s_i} and T_{s_j} commute, and p_i,p_j also commute:

$$p_i p_j = p_j p_i. \tag{2}$$

In TL_q^c we have relation $1 + T_{s_i} + T_{s_j} + T_{s_i}T_{s_j} + T_{s_j}T_{s_i} = 0$, which is equivalent to

$$p_i p_j = p_j p_i = 0. \tag{3}$$

If $s_i s_j$ has order 3 in W, then $T_{s_i}T_{s_j}T_{s_i} = T_{s_j}T_{s_i}T_{s_j}$ in H_q, which is equivalent to

$$p_i p_j p_i - \tau p_i = p_j p_i p_j - \tau p_j. \tag{4}$$

Then in TL_q and TL_q^c there is a relation $1 + T_{s_i} + T_{s_j} + T_{s_i}T_{s_j} + T_{s_j}T_{s_i} + T_{s_i}T_{s_j}T_{s_i} = 0$, which is equivalent to

$$p_i p_j p_i - \tau p_i = p_j p_i p_j - \tau p_j = 0. \tag{5}$$

As it can be seen from the definitions, H_q, TL_q^c and TL_q can be presented via generators T_{s_i}, $s_i \in S$ with relations mentioned above. So, we get the following presentation

Proposition 1. *$H_q(W)$ is generated by idempotents $p_1, \dots, p_n$ with relations (4) for each pair of noncommuting generators $s_i, s_j \in W$ and (2) for each pair of commuting generators $s_i, s_j \in W$.*

$TL_q(W)$ is its quotient by relations (5) for each pair of noncommuting generators $s_i, s_j \in W$.

$TL_q^c(W)$ is a quotient of $TL_q(W)$ by relations (3) for each pair of commuting generators $s_i, s_j \in W$.

In [8] and [9] they study $*$-representations of algebras, generated by projections, which are exactly TL_q^c of Coxeter groups of type A_n and $\widetilde{A_n}$ correspondingly.

In work [10] of the author some generalizations of TL_q^c are considered, where parameter τ in (5) may depend on i, j. It is noted that relations in presentation of TL_q^c by idempotents from Proposition 1 form a Groebner basis (where ordering on generators p_i can be taken arbitrarily, and we consider corresponding homogenous lexicographical ordering on their noncommutative monomials). In other words, monomials in letters $p_1, \ldots, p_n$, which does not contain the following submonomials

$$p_ip_jp_i \text{ and } p_jp_ip_j \text{ for each non-commuting } s_i, s_j \in W,$$
$$p_ip_j \text{ and } p_jp_i \text{ for each commuting } s_i, s_j \in W$$

together with $\mathbf{1}$ form a linear basis of TL_q^c. Consider the Coxeter graph Γ of Coxeter group W. As its vertices correspond to $s_i \in S$, they also correspond to $\{p_i, 1 \le i \le n\}$. So, all elements of this linear basis but $\mathbf{1}$ can be identified with paths in Γ with one restriction: we consider only such paths in which all neighboring edges differ. So, we get the following

Proposition 2. *An algebra $TL_q^c(W)$ is finite-dimensional iff Coxeter graph Γ of W does not contain cycles. The dimension of TL_q^c is*

$$\dim(TL_q^c) = 1 + \sum_{\Gamma' \in \pi_0(\Gamma)} |\Gamma'|^2,$$

where $\pi_0(\Gamma)$ is a set of connected components and $|\Gamma'|$ is a number of vertices in connected component Γ'.

If Γ contains not more than one cycle in each connected component, then the growth of an algebra TL_q^c is linear, i.e. its Gelfand–Kirillov dimension equals 1.

If some connected component of Γ contains more than one cycle, an algebra TL_q^c contains a free subalgebra on two generators.

3 The structure of finite-dimensional algebras TL_q^c

In this section we are going to prove

Theorem 1. *Let Coxeter graph Γ of group W do not contain cycles. Then for almost all $q \in \mathbb{C}$ (except a finite set)*

$$TL_q^c \cong M_1(\mathbb{C}) \bigoplus_{\Gamma' \in \pi_0(\Gamma)} M_{|\Gamma'|}(\mathbb{C}).$$

Remark 2. There are values of q, for which TL_q^c is not semisimple. For example, $TL^s_{\frac{1+\sqrt{-3}}{2}}(S_3)$ is not semisimple as it coincides with $TL_{\frac{1+\sqrt{-3}}{2}}(S_3)$.

Let Γ be a tree with n vertices, and Γ_0 be the set of its vertices. One can take any vertex $\lambda \in \Gamma$ as a root. Then binary relation of being a child is defined as usually on vertices of a rooted tree: $\beta \in \Gamma_0$ is a child of $\alpha \in \Gamma_0$ iff there is an edge between α and β and β is farther from the root λ then α. Define $c_\lambda : \Gamma_0 \longrightarrow 2^{\Gamma_0}$ so that $c_\lambda(\alpha)$ is a set of children vertices of α.

We call $\lambda \in \Gamma_0$ a *matching* vertex if there exists a function $a : \Gamma_0 \longrightarrow \mathbb{C}\backslash\{0\}$ such that

$$a(\alpha) + \tau \sum_{\beta \in c_\lambda(\alpha)} \frac{1}{a(\beta)} = 1 \text{ for each } \alpha \in \Gamma_0, \tag{6}$$

where $\tau = \frac{q}{(q+1)^2}$.

Lemma 1. *If there is a* matching *vertex in graph Γ, then there is an irreducible representation of TL_q^c in dimension $n = |\Gamma|$, in which all generating idempotents are nonzero.*

Proof. We give an explicit construction of such a representation. Consider a vector space V with basis $\{e_\beta; \beta \in \Gamma_0\}$. Let λ be a *matching* vertex. Let $a_\beta = \sqrt{a(\beta)}$, where $a : \Gamma_0 \longrightarrow \mathbb{C}\backslash\{0\}$ be a function from condition (6) above. Let $g_\beta = a_\beta e_\beta + \sum_{\alpha \in c_\lambda(\beta)} \frac{\sqrt{\tau}}{a_\alpha} e_\alpha$. Consider the following action of generating idempotents:

$$p_\beta e_\mu = \begin{cases} a_\beta g_\beta, & \mu = \beta; \\ \dfrac{\sqrt{\tau}}{a_\mu} g_\beta, & \mu \in c_\lambda(\beta); \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (6) that $p_\beta g_\beta = g_\beta$. If vertices α and β are not connected in Γ, then also $c_\lambda(\alpha) \cap c_\lambda(\beta) = \varnothing$, so $p_\beta g_\alpha = p_\alpha g_\beta = 0$. If vertices α and β are connected, then one has $p_\beta g_\alpha = \sqrt{\tau} g_\beta$. So, given above is a representation.

To prove it is irreducible we consider matrix units acting on V:

$$E_{\alpha,\beta} e_\mu = \begin{cases} e_\alpha, & \mu = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

If β is a leaf then $p_\beta = E_{\beta,\beta}$. Suppose that for some α matrix units $E_{\beta,\beta}$ are defined for all $\beta \in c_\lambda(\alpha)$ via $\{p_\mu; \mu \in \Gamma_0\}$. Then

$$E_{\alpha,\alpha} = \frac{1}{a_\alpha^2}\left(1 - \sum_{\beta \in c_\lambda(\alpha)} E_{\beta,\beta}\right) p_\alpha \left(1 - \sum_{\beta \in c_\lambda(\alpha)} E_{\beta,\beta}\right)$$

also is specified. And for each $\beta \in c_\lambda(\alpha)$

$$E_{\beta,\alpha} = \frac{a_\beta}{\sqrt{\tau} a_\alpha} E_{\beta,\beta} p_\alpha E_{\alpha,\alpha}, \qquad E_{\alpha,\beta} = \frac{a_\beta}{\sqrt{\tau} a_\alpha} E_{\alpha,\alpha} p_\alpha E_{\beta,\beta}.$$

Thus an image of TL_q^c in $\mathrm{End}_{\mathbb{C}}(V)$ contains $E_{\alpha,\alpha}$ for all $\alpha \in \Gamma_0$ and $E_{\alpha,\beta}$ for all pairs of connected vertices α and β. So representation constructed above is irreducible as graph Γ is connected. ■

Note that it is easy to verify if some $\lambda \in \Gamma_0$ is *matching*: the values $a(\alpha)$ for $\alpha \in \Gamma_0$ from (6) are defined uniquely by λ. To find them one should put $a(\alpha) = 1$ for all leafs α, and then calculate all other values by recursive formulae (6).

Remark 3. A tree Γ has a *matching* vertex for almost all $q \in \mathbb{C}$. Indeed, when arbitrary vertex λ is taken to be a root, we can define $a(\alpha)$ from (6) with values in $\mathbb{C}(q)$. Then numerators of irreducible fractions $a(\alpha) \in \mathbb{C}(q)$ is a set of certain polynomials $\{P_\alpha^\lambda(q); \alpha \in \Gamma_0\}$, and λ is a *matching* vertex iff $P_\alpha^\lambda(q) \neq 0$ for all $\alpha \in \Gamma_0$. An intersection of this finite sets for all $\lambda \in \Gamma_0$ is exactly the set of such $q \in \mathbb{C}$, which we exclude in Theorem 1 in case of connected Γ.

Now we are going to prove Theorem 1.

Proof. Due to Remark 3 for almost all $q \in \mathbb{C}$ all connected components $\Gamma' \in \pi_0(\Gamma)$ have *matching* vertices. For such a q we can construct a set of irreducible representations $\{V^{\Gamma'}; \Gamma' \in \pi_0(\Gamma)\}$ one for each connected component. Indeed, for each Γ' we can take a representation of corresponding subalgebra from Lemma 1, and induce representation of TL_q^c putting all generating idempotents from other connected components to be zero. Moreover, in this construction representations corresponding to different connected components are nonisomorfic.

There is one more 1-dimensional representation – the trivial one, in which all generating idempotents are zero. It is nonisomorfic to any one from a list $\left\{V^{\Gamma'}; \Gamma' \in \pi_0(\Gamma)\right\}$. So, we have a list of pairwise nonisomorfic irreducible representations, and due to Proposition 2

$$\dim(TL_q^c) = \dim(\mathbb{C})^2 + \sum_{\Gamma' \in \pi_0(\Gamma)} \dim\left(V^{\Gamma'}\right)^2.$$

This implies

$$TL_q^c \cong M_1(\mathbb{C}) \bigoplus_{\Gamma' \in \pi_0(\Gamma)} M_{|\Gamma'|}(\mathbb{C})$$

by Jacobson's density theorem. ■

[1] Humphreys J.E., Reflection groups and Coxeter groups, Cambridge, Cambridge University Presss, 1990.

[2] Barcelo H. and Ram A., Combinatorial representation theory, MSRI, 1997.

[3] Green R.M. and Losonczy J., Canonical bases for Hecke algebra quotient, *Math. Res. Lett.*, 1999, V.6, 213–222.

[4] Goldshmidt D.M., Group characters, symmetric functions and the Hecke algebra, *University Lecture Series*, Vol. 4, Providence, R.I., AMS, 1993.

[5] Goodman F.M. and Wenzl H., The Temperley–Lieb algebra at roots of unity, *Pacific Journ. Math.*, 1993, V.161, N 2, 307–333.

[6] Graham J.J., Modular representations of Hecke algebras and related algebras, PhD Thesis, University of Sydney, 1995.

[7] Ufnarovskiĭ V., Combinatorical and asimptotic methods in algebra, *Current Problems in Mathematics. Fundamental Directions*, Vol. 57, Moscow, VINITI, 1990, 5–177 (in Russian).

[8] Wenzl H., On sequences of projections, *C.R. Math. Rep. Acad. Sci. Canada*, 1987, V.9, N 1, 5–9.

[9] Popova N., On one algebra of Temperley–Lieb type, in Proceedings of Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–15 July, 2001, Kyiv), Editors A.G. Nikitin, V.M. Boyko and R.O. Popovych, Kyiv, Insitute of Mathematics, 2002, V.43, Part 2, 486–489.

[10] Vlasenko M., On the growth of algebra established by a system of projections with fixed angles, *Meth. Func. Anal. Topology*, 2004, to appear.