Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 3, 1510–1517

On the Localized Invariant Solutions of Some Non-Local Hydrodynamic-Type Models

Vsevolod A. VLADIMIROV and Ekaterina V. KUTAFINA

Faculty of Applied Mathematics, University of Science and Technology,
30 Mickiewicz Ave., 30-059 Kraków, Poland
E-mail: *vsan@rambler.ru, vladimir@mat.agh.edu.pl*

We analyze the conditions, assuring the existence of invariant wave patterns in the non-local hydrodynamic models of structured media.

1 Introduction

This paper deals with a family of invariant solutions of some modelling systems of PDE, taking into account non-local effects. These effects are manifested when an intense pulse loading (impact, explosion etc.) is applied to media, possessing an internal structure on mesoscale. Description of the non-linear waves propagation in such media depends in essential way on the ratio of a characteristic size d of elements of the medium structure to a characteristic length λ of the wave pack. If d/λ is $O(1)$, then the basic concepts of continuum mechanics are not applicable any more, and one should use the description, based, e.g. on the element dynamics methods [1]. The applications of classical continuum mechanics equations are justified in those cases, when $d/\lambda \ll 1$, and the discreteness of the matter could be completely ignored.

The models studied in this work apply when the ratio d/λ is much less that unity and therefore the continual approach is still valid, but it is not as small that we can ignore the presence of the internal structure. As it has been shown in a number of papers (see e.g. [2]), in the long wave approximation the balance equations for mass and momentum retain their classical form, which in the one-dimensional case can be written as follows:

$$u_t + p_x = 0, \qquad \rho_t + \rho^2 u_x = 0, \tag{1}$$

where u is the mass velocity, p is the pressure, ρ is the density, t is the time, x is the mass (Lagrangian) coordinate, lower indices denote partial derivatives with respect to subsequent variables. Thus, the whole information about the presence of structure in this approximation is contained in a dynamic equation of state (DES), which should be incorporated to system (1) in order to make it closed.

Generally speaking, DES take on the form of integral equations [3], linking the thermodynamical flows I_n and generalized thermodynamical forces L_m, causing these flows:

$$I_n = \int_{-\infty}^{t} dt' \int_R dx' K_{nm}(t,t',x,x') L_m(t',x'). \tag{2}$$

Here $K_{nm}(t,t',x,x')$ is a kernel, taking into account nonlocal effects. Function K_{nm} can be calculated, in principle, by solving dynamic problem of structure's elements interaction, however such calculations are extremely difficult. Therefore in practice one uses, as a role, some model kernels, describing well enough the main properties of the non-local effects and, in particular, the fact that these effects vanish rapidly as $|t-t'|$ and $|x-x'|$ grow. This property will enable us to pass from the equations of the integro-differential type to systems of differential equations, including higher-order derivatives.

Thus our goal is to present the modelling systems, associated with the integral DES, which describe the non-local effects connected with structure. We show that, in spite of the fact that the systems of PDE's considered here are not Hamiltonian, they do possess families of invariant solutions, satisfying certain systems of ODE's, being equivalent to the Hamiltonian ones. Due to this property, it becomes possible to state the existence of periodic and soliton-like solutions, which are inherent to the models of structured media and presumably do not occur in hydrodynamic-type models of structureless media. At least, that is true under the assumption that the characteristic velocity of the wave perturbations is a growing function of ρ.

2 Invariant localized solutions in the model with spatial nonlocality

One of the simplest state equation accounting for the effects of spatial nonlocality takes on the form [4]:

$$p = \hat{\sigma} \int_{-\infty}^{+\infty} \rho^n(t, x') \exp\left\{ - \left[(x - x') / l \right]^2 \right\} dx'. \tag{3}$$

Since the kernel $K(x, x') = \exp\left\{ - \left[(x - x') / l \right]^2 \right\}$ extremely quickly approaches zero as $|x - x'|$ grows, we can substitute the function $\rho^n(t, x')$ by its decomposition into the power series:

$$\rho^n(t, x') = \rho^n(t, x) + [\rho^n(t, x)]_x \frac{x' - x}{1!} + [\rho^n(t, x)]_{xx} \frac{(x' - x)^2}{2!} + o\left(|x - x'|^2 \right),$$

obtaining this way so called gradient model [4]:

$$p = c_0 \hat{\sigma} \rho^n(t, x) + c_2 \hat{\sigma} [\rho^n(t, x)]_{xx}. \tag{4}$$

Here

$$c_0 = l \int_{-\infty}^{+\infty} e^{-\tau^2} d\tau = l\sqrt{\pi}, \qquad c_2 = \frac{l^3}{2} \int_{-\infty}^{+\infty} \tau^2 e^{-\tau^2} d\tau = \frac{l^3\sqrt{\pi}}{4}.$$

Inserting (4) into the first equation of system (1), we obtain a closed system:

$$\begin{aligned} &u_t + \beta \rho^{\nu+1} \rho_x + \sigma \left[\rho^{\nu+1} \rho_{xxx} + 3(1+\nu) \rho^\nu \rho_x \rho_{xx} + \nu(1+\nu) \rho^{\nu-1} \rho_x^3 \right] = 0, \\ &\rho_t + \rho^2 u_x = 0, \end{aligned} \tag{5}$$

where $n = \nu + 2$, $\beta = c_0 \hat{\sigma}(\nu + 2)$, $\sigma = c_2 \hat{\sigma}(\nu + 2)$. Below we analyze a family of invariant travelling wave solutions

$$u = U(\omega), \qquad \rho = R(\omega), \qquad \omega = x - Dt. \tag{6}$$

Inserting the anzatz (6) into the second equation of system (5), we obtain the first integral:

$$U = C_1 - D/R. \tag{7}$$

Asymptotic conditions $\lim\limits_{\omega \to +\infty} U(\omega) = 0$, $\lim\limits_{\omega \to +\infty} R(\omega) = R_1 > 0$, which we assume to take place further on, get the expression $C_1 = D/R_1$. Substituting (6) into the second equation of system (5) and using the formula (7), we obtain a third-order equation for function $R(\omega)$. This equation can also be integrated, and, having done that, we finally get the second order ODE

$$\frac{D^2}{R} + \frac{\beta}{\nu + 2} R^{\nu+2} + \sigma \left[R^{\nu+1} \frac{d^2 R}{d\omega^2} + (\nu + 1) R^\nu \left[\frac{dR}{d\omega} \right]^2 \right] = E, \tag{8}$$

where

$$E = \frac{D^2}{R_1} + \frac{\beta}{\nu+2} R_1^{\nu+2}, \tag{9}$$

is a constant of integration, defined by the conditions on $+\infty$.

Let us write equation (8) in the form of the first order dynamic system:

$$\begin{aligned} \frac{dR}{d\omega} &= Y, \\ \frac{dY}{d\omega} &= \left(\sigma R^{\nu+2}\right)^{-1} \left\{ ER - \left[D^2 + \frac{\beta}{\nu+2} R^{\nu+3} + \sigma(\nu+1) R^{\nu+1} Y^2 \right] \right\}. \end{aligned} \tag{10}$$

It is evident that all isolated critical points of system (10) are located on the horizontal axis OR. They are determined by solutions of the algebraic equation

$$P(R) = \frac{\beta}{\nu+2} R^{\nu+3} - ER + D^2 = 0. \tag{11}$$

As can be easily seen, one of the roots of equation (11) coincides with R_1. Location of the second real root depends on relations between the parameters. If $\nu+3>1$ and D^2 satisfies inequality

$$D^2 > D^2_{\text{cr}} = \beta R_1^{\nu+3}, \tag{12}$$

then there exists the second critical point $R_2 > R_1$. Moreover, if $\nu > 0$ is a natural number, then the polynomial $P(R)$ has the representation

$$P(R) = (R - R_1)(R - R_2)\Psi(R), \tag{13}$$

where

$$\Psi(R) = \frac{\beta}{(\nu+2)(R_2 - R_1)} \left\{ R^{\nu+1} + R^{\nu}(R_2 - R_1) + \cdots + R\left(R_2^{\nu} - R_1^{\nu}\right) + \left(R_2^{\nu+1} - R_1^{\nu+1}\right) \right\}.$$

Note that $\Psi(R)$ is positive, when $R > 0$. It is a direct consequence of the existence of representation (13) when ν is a natural number. But this is also true for any $\nu > -2$, or, in other words, whenever the function $R^{\nu+3}$ is concave for positive R.

Analysis of linearization matrix for the system (10)

$$\hat{M}(R_i, 0) = \begin{bmatrix} 0 & 1 \\ \left(\sigma R_i^{\nu+2}\right)^{-1} \Psi(R_i)(R_j - R_i) & 0 \end{bmatrix}, \qquad i = 1, 2, \qquad j \neq i \tag{14}$$

shows that the critical points $A_1(R_1, 0)$ is a saddle, while the critical point $A_2(R_2, 0)$ is a center. Thus, system (10) has only such critical points that are characteristic to the Hamiltonian system. This circumstance suggests that there could exist a Hamiltonian system equivalent to (10). The Hamiltonian function would help us to make a complete study of the phase portrait of system (10) and find out homoclinic trajectories, which correspond to a soliton-like wave packs, providing that such trajectories do exist.

If we introduce a new independent variable T, obeying the equation $\frac{d}{dT} = \sigma R^{\nu+2}\phi(R,Y)\frac{d}{d\omega}$, then system (10) can be written as

$$\begin{aligned} \frac{dR}{dT} &= \sigma R^{\nu+2} \phi(R, Y), \\ \frac{dY}{dT} &= \phi(R, Y) \left(ER - \left[D^2 + \frac{\beta}{\nu+2} R^{\nu+3} + \sigma(\nu+1) R^{\nu+1} Y^2 \right] \right). \end{aligned} \tag{15}$$

Here $\phi(R,Y)$ is a function, that is to be chosen in such a way that system (15) be Hamiltonian. So we assume the existence of a function $H(R,Y)$, satisfying the system

$$\frac{\partial H}{\partial Y} = \sigma R^{\nu+2}\phi(R,Y),$$
$$\frac{\partial H}{\partial R} = -\phi(R,Y)\left\{ER - \left[D^2 + \frac{\beta}{\nu+2}R^{\nu+3} + \sigma(\nu+1)R^{\nu+1}Y^2\right]\right\}.$$

Equating mixed derivatives of H, we obtain the characteristic system

$$\frac{dR}{\sigma R^{\nu+2}Y} = \frac{dY}{ER - [D^2 + \frac{\beta}{\nu+2}R^{\nu+3} + \sigma(\nu+1)R^{\nu+1}Y^2]} = \frac{d\phi}{\nu\sigma Y R^{\nu+1}\phi}. \tag{16}$$

The general solution of system (16) can be written as follows:

$$\phi = R^\nu \Psi(\Omega),$$

where $\Psi(\cdot)$ is an arbitrary function of the variable

$$\Omega = R^{\nu+1}\left\{Y^2 + D^2\sigma^{-1}R^{-(\nu+1)}\ln R + R^2\beta/[\sigma(\nu+2)(\nu+3)] - E\sigma^{-1}R^{-\nu}\right\}.$$

Putting $\phi = 2R^\nu$, we can easily restore the Hamiltonian function:

$$H = 2D^2\frac{R^{\nu+1}}{\nu+1} + \frac{\beta}{(\nu+2)^2}R^{2(\nu+2)} + \sigma Y^2 R^{2(\nu+1)} - 2E\frac{R^{\nu+2}}{\nu+2}. \tag{17}$$

By elementary checking one can get convinced that the function H is constant on phase trajectories of both systems (10) and (15), and since the integrating multiplier $\phi = 2R^\nu$, occurring in formula (15) is positive for $R > 0$, then phase trajectories of systems (10) and (15) are almost similar in the right half-plane of the phase plane (R,Y), the only difference between them manifesting in velocities of motion. Thus all the statements concerning the geometry of the phase trajectories of system (15) lying in the right half-plane is applicable to corresponding solutions of system (10).

Using the linear analysis we showed that the critical point $A_2(R_2,0)$ is a center. Stated above relations between systems (10) and (15) enable to conclude that this point does not change when the nonlinear terms are added. This means that the critical point $A_2(R_2,0)$ is surrounded by closed trajectories and hence the source system (5) possesses a one-parameter family of periodic solutions. If the right branches of the separatrices of the saddle $A_1(R_1,0)$ go to infinity (the stable branch W^s when $t \to -\infty$ and the unstable branch W^u when $t \to +\infty$), then the domain of finite periodic motions is unlimited. Another possibility is connected with the existence of homoclinic trajectories in phase space. In this case the domain of periodic solutions is bounded, and the source system, besides periodic solutions, possesses localized soliton-like regimes. To answer the question on which of the above mentioned possibilities is realised in system (10), the behavior of the saddle separatices, lying to the right from the line $R = R_1$ should be analyzed. We obtain the equation for saddle separatices by putting $H = H(R_1,0) = H_1$ in the left hand side of the equation (17) and solving it next with respect to Y:

$$Y = \pm\frac{\sqrt{H_1 + 2E\frac{R^{\nu+2}}{\nu+2} - \left[2D^2\frac{R^{\nu+1}}{\nu+1} + \frac{\beta}{(\nu+2)^2}R^{2(\nu+2)}\right]}}{\sqrt{\sigma}R^{\nu+1}}. \tag{18}$$

It is evident from equation (18), that incoming and outgoing separatrices are symmetrical with respect to OR axis. Therefore we can restrict our analysis to one of them, e.g. to the upper separatrix Y_+. First of all, let us note, that in the point$(R_1, 0)$ separatrix Y_+ forms with OR axis a positive angle

$$\alpha = \arctan \sqrt{(R_2 - R_1)\Psi(R_1)/\left(\sigma R_1^{\nu+2}\right)}.$$

The above formula arises from the linear analysis of system (10) in critical point $A_1(R_1, 0)$. So $Y_+(R)$ is increasing when $R - R_1$ is small and positive. On the other hand, function

$$G(R) = H_1 + 2E\frac{R^{\nu+2}}{\nu+2} - \left[2D^2\frac{R^{\nu+1}}{\nu+1} + \frac{\beta}{(\nu+2)^2}R^{2(\nu+2)}\right],$$

standing inside the square root in equation (18), tends to $-\infty$ as $R \to +\infty$, because the coefficient at the highest order monomial $R^{2(\nu+2)}$ is negative, while the index $\nu + 2$ is assumed to be positive. Therefore the function $G(R)$ intersects the open set $R > R_1$ of the OR axis at least once. Let us denote a point of the first intersection by R_3, and let us assume that $R_3 > R_2$. If with this assumption we were able to prove that $Y_\pm(R)$ form the right angle with the OR axis at the point R_3, then we would have the evidence of the continuous homoclinic loop existence.

We begin with the note that $\lim_{R\to R_3^-} G(R) = +0$. Calculating derivative of $G(R)$ we have:

$$G'(R) = -2R^\nu\left(\frac{\beta}{\nu+2}R^{\nu+3} - ER + D^2\right) = -2R^\nu P(R). \tag{19}$$

It follows from the decomposition (13) that $G'(R) < 0$ when $R > R_2$. Therefore

$$\lim_{R\to R_3^-}\frac{dY}{dR} = \frac{RG'(R) - 2(\nu+1)G(R)}{2\sqrt{\sigma G}R^{\nu+2}} = -\infty$$

and we merely have to show that the inequality $R_3 > R_2$ is true. Supposing that the inequalities $R_1 < R_3 < R_2$ take place, we obtain from equations (19), (13) that $\lim\limits_{R\to R_3^-} Y_+'(R) = +\infty$. On the other hand, the function $Y_+(R)$ approaches zero remaining positive as $R \to R_3^-$. But such behavior is impossible for any function, regular on the interval (R_1, R_3). The case $R_3 = R_2$ should also be excluded, because the critical point $A_2(R_2, 0)$ is a center. The result obtained can be formulated as follows.

Theorem 1. *If $\nu > -2$ and $D^2 > \beta R_1^{\nu+3}$, then system (10) possesses a one parameter family of periodic solutions, localized around the critical point $A_2(R_2, 0)$ in a bounded set* **M**. *The boundary of this set is formed by the homoclinic intersection of separatrices of the saddle point $A_1(R_1, 0)$.*

Thus the source system (5) possesses periodic and soliton-like invariant solutions.

3 Invariant wave patterns in the model with spatio-temporal nonlocality

In this section we consider system (1), closed by DES:

$$p = \int_{-\infty}^{t} sin\frac{t-t'}{\tau_1}e^{-\frac{t-t'}{\tau_2}}L[\rho, \rho_x, \rho_{xx}]dt', \tag{20}$$

where

$$L[\rho, \rho_x, \rho_{xx}] = \frac{\beta}{\nu+2}\rho^{\nu+2}(t', x) + \hat{\sigma}[\rho^{\mu+1}(t', x)\rho_x(t', x)]_x.$$

Taking the second derivative of (20) with respect to t, we obtain a higher-order equation, which, together with (1), forms a closed system of the following form:

$$u_t + p_x = 0, \qquad \rho_t + \rho^2 u_x = 0,$$
$$h p_{tt} + \tau p_t + p = \frac{\beta}{\nu+2}\rho^{\nu+2} + \sigma\left[\rho^{\mu+1}\rho_{xx} + (\mu+1)\rho^{\mu}(\rho_x)^2\right], \tag{21}$$

where $h = (\tau_1\tau_2)^2/\left(\tau_1^2+\tau_2^2\right)$, $\tau = 2\tau_1^2\tau_2/\left(\tau_1^2+\tau_2^2\right)$.

As in the previous section, we analyze the family of invariant travelling wave solutions

$$u = U(\omega), \qquad \rho = R(\omega), \qquad p = \Pi(\omega), \qquad \omega = x - Dt. \tag{22}$$

Inserting (22) into (21), we obtain a system of ODE, possessing two integrals. The first one, arising from the balance of mass equation, coincides with (7), while the second one, arising from the momentum equation, is as follows:

$$\Pi = E - D^2/R. \tag{23}$$

If the conditions on $+\infty$ are identical with those from the previous section, then the expression for E is given by the formula (9). Taking advantage of the formulae (7), (23), we can write down the remaining second order equation in the form of a dynamic system:

$$R\Delta\phi\dot{R} = YR\Delta\phi,$$
$$R\Delta\phi\dot{Y} = -\phi\left\{R^2P(R) + Y^2\left[2hD^4 + \sigma(\mu+1)R^{\mu+3}\right] + \tau D^3R^2Y\Delta\right\}, \tag{24}$$

where $\Delta = \sigma R^{\mu+3} - hD^4$ and $P(R)$ is given by the formula (13). Here we introduced an extra function $\phi(R,Y)$ that would help us to make system (24) Hamiltonian.

We postpone with the construction of the Hamiltonian function for system (24) and study its critical points first, for a pure linear analysis shows that the multiplier $\phi(R,Y)$ must have the singularities in the area of interest, unless some parameters nullify. It is evident that all isolated critical points of system (24) satisfy the equation (11). The case we are interested in is again that with two real solutions R_1, R_2, satisfying the inequality $R_2 > R_1$. Therefore we assume that parameter D^2 satisfies the inequality (12). An extra inequality

$$\frac{\beta(\nu+3)}{\nu+2}R_1^{\nu+2} < E < \frac{\beta(\nu+3)}{\nu+2}R_2^{\nu+2}, \tag{25}$$

which will be used later on, is the direct consequence of the inequality (12) and concavity of the function $\beta R^{\nu+3}$.

The linearization matrix of system (24) in a critical point $A_i\,(R_i, 0)$, $i = 1, 2$, takes on the form:

$$\hat{M}(R_i,0) = \begin{bmatrix} 0, & R_i\Delta_i\phi(R_i,0) \\ \phi(R_i,0)R_i\left(3ER_i - 2D^2 - \frac{\beta(\nu+5)}{\nu+2}R_i^{\nu+3}\right), & -\phi(R_i,0)\tau D^3R_i^2\Delta_i \end{bmatrix}, \tag{26}$$

where $\Delta_i = \sigma R_i^{\mu+3} - hD^4$. Thus, the eigenvalues of matrix $\hat{M}(R_i,0)$ satisfy the equation

$$\lambda\left(\lambda + \phi(R_i,0)\tau D^3R_i^2\Delta_i\right) = \phi(R_i,0)^2\Delta_iR_i^3\left[E - (\nu+3)(\nu+2)^{-1}R^{\nu+2}\right]. \tag{27}$$

Considering equation (27) together with the inequalities (25) one can easily conclude that the features of the isolated critical points of system (24) depend in essential way on the sign of Δ_i. If R_1 is placed to the right from the line $\Delta(R) = 0$, i.e.

$$R_4 = \left(D^4h/\sigma\right)^{1/(\mu+3)} < R_1 \tag{28}$$

and therefore inequality $\Delta(R_i) > 0$ holds for both $i = 1, 2$, then the eigenvalues corresponding to critical point $A_1\,(R_1, 0)$ are nonzero and have the opposite signs, so this point still is a saddle. The critical point $A_2\,(R_2, 0)$ turns out to be either a stable focus, when τ is sufficiently small, or even a stable node. Anyhow, it is rather impossible to chose a nonsingular multiplier ϕ, unless $\tau = 0$ and the critical point $A_2\,(R_2, 0)$ is a center.

If the inequality $R_4 > R_2$ takes place, then the critical point $A_2\,(R_2, 0)$ is a saddle, while $A_1\,(R_1, 0)$ becomes a focus. The situation is more complicated when the inequalities $R_1 < R_4 < R_2$ hold, but in our attempts to capture the homoclinic trajectories we will rather focus upon the case $R_4 < R_1$, as the only one leading to the localized compressed waves appearance. Before we start to study the Hamiltonian case, let us pay attention to the fact that the inequalities (25), together with the condition $R_4 < R_1$, pose the following restrictions on the invariant wave pack velocity D:

$$\beta R_1^{\nu+3} < D^2 < \sqrt{\sigma h^{-1} R_1^{\mu+3}} \tag{29}$$

As we have mentioned above, it is rather impossible to chose a proper multiplier ϕ, whenever $\tau \neq 0$, because the corresponding term in the RHS of the second equation of system (24) introduces the dissipation. Yet, by proper choice of the parameters τ_1 and τ_2 in equation (20), we can fulfil the relations $\tau \ll 1$ and $h = O(1)$, enabling to drop down the term proportional to τ. For $\tau = 0$ function ϕ can be written, e.g. in the form $\phi = C\Delta R^{-5}$, where C is an arbitrary constant. The system (24) becomes Hamiltonian, if we choose ϕ in accordance with the above formula and introduce new independent variable T, using the relation $R\Delta\phi\frac{d}{dT} = \frac{d}{d\omega}$.

With such a choice of the multiplier, the Hamiltonian function is as follows:

$$\begin{aligned} H(R, Y) = \frac{\Delta^2}{\sigma R^4} Y^2 + \frac{1}{\sigma R^2} \Bigg\{ D^6 h + \frac{2D^2 \sigma R^{\mu+3}}{\mu+1} - \frac{2E\sigma}{\mu+2} R^{\mu+4} \\ + \frac{2\sigma\beta}{(\nu+2)(\nu+\mu+4)} R^{\nu+\mu+6} - 2D^4 hRE - \frac{2D^4 h\beta R}{\nu^2+3\nu+2} \Bigg\}. \end{aligned} \tag{30}$$

To show that the closed loop does exist among the solutions of system (24), let us consider one of the separatrices of the saddle point $A_1\,(R_1, 0)$, namely:

$$Y_+(R) = \sqrt{\frac{\tilde{G}(R)}{F(R)}}, \tag{31}$$

where

$$\tilde{G}(R) = H\,(R_1, 0) - H\,(R, 0)\,, \qquad F(R) = \frac{\Delta^2}{R^4\sigma} \tag{32}$$

(note, that both equation (30) and (31), become identical with the corresponding equations of the previous section as we put $h = 0$ and $\nu = \mu$). As it follows from the linear analysis of system (24) in the critical point $A_1\,(R_1, 0)$, the curve $Y_+(R)$ forms with OR axis a positive angle α:

$$\alpha = \arctan\sqrt{{R_1}^2{\Delta_1}^{-1}\left[E - (\nu+3)(\nu+2)^{-1}R_1^{\nu+2}\right]}.$$

In virtue of the function $\tilde{G}(R)$ features, arising from the formulae (30), (32), it should intersect open set $(R_1, +\infty)$ at least once. Let us denote the point of the first intersection by R_3. Repeating the analysis from Section 2, we can show that the saddle separatrix $Y_+(R)$ and the OR axis form the right angle in the point of intersection and the inequalities $R_3 > R_2 > R_1$ take place. So the following statement holds.

Theorem 2. *If $\nu > -2$, $R_1 > \left(D^4 h/\sigma\right)^{1/(\mu+3)}$ and D^2 satisfies inequalities* (29)*, then system* (24) *possesses a one parameter family of periodic solutions, localized around the critical point $A_2\,(R_2, 0)$ in a bounded set* **M**. *The boundary of this set is formed by the homoclinic intersection of separatrices of the saddle point $A_1\,(R_1, 0)$.*

4 Conclusions

Two non-local models have been presented describing long waves propagation in the media with internal structure. The main result obtained is that the hydrodynamic-type systems, accounting for non-local effects, possess periodic and soliton-like travelling wave solutions. The presence of the above solutions seems to be the direct consequence of the non-local effects, since any local hydrodynamic-type model, i.e. the system of balance equations (1) closed by the functional state equation $p = \Phi(\rho)$, does not possess them, when $d\Phi(\rho)/d\rho$ is a growing function for positive ρ. Numerous counter-examples (given e.g. in [5]) do not satisfy this condition and therefore deliver merely a set of unstable travelling wave solutions.

In order to prove the existence of nonlinear wave patterns we used in this work combination of symmetry reduction and qualitative analysis. Let us stress that a number of solutions with very interesting applications cannot be integrated explicitly, and the role of qualitative analysis is hardly to be overestimated in such situations. Very often they do enable to prove the existence of solutions of specific kind. For example the soliton-like solutions could be extracted by regular methods, providing that the system under consideration is either Hamiltonian, or close to the Hamiltonian [6]. But the preliminary studies of system (5) show [7], that it is not Hamiltonian for any physically justified values of the parameters. Nevertheless, the dynamic system, obtained from (5) by group-theory factorization, occurs to be equivalent to the Hamiltonian one, and therefore it is relatively easy to analyze. System (21), in turn, is a non-conservative system, because of the presence of dissipative term proportional to τ in the third (governing) equation. Therefore soliton-like solutions belonging to the family (22) appear when τ nullifies. Strictly speaking, the passage $\tau \to 0$ is an improper procedure, but the results obtained in the last section might serve as a starting point in looking for the waves patterns in the case when $0 < \tau \ll 1$ and an external force is present in the momentum equation. Analysis of the relaxing hydrodynamic-type system, undertaken in our previous works, shows [8] that the spatial inhomogeneity, associated with the presence of mass force, could compensate the dissipative effects, making possible the wave patterns appearance.

[1] Danilenko V.A., Belinskij I.V. and Vengrovich D.B., On the peculiarities of the wave processes connected with the structure of geophysical media, *Dopovidi NAS of Ukraine*, 1996, N 12, 124–129.

[2] Vakhnenko V.A. and Kulich V., Averaged equations of wave dynamics of the periodic relaxing media, in Boundary Value Problems, Editor V.A. Danylenko, Kyiv, Naukova Dumka, 1990, 118–120.

[3] Zubarev D. and Tishchenko S., Nonlocal hydrodynamics with memory, *Physica*, 1972, V.50, N 2, 285–304.

[4] Peerlings R.H.J., Enhanced damage modelling for fracture and fatigue, Eindhoven, Technische Universiteit Eindhoven, 1999.

[5] Zhdanov S.K. and Trubnikov B.F., Unstable quasi-gaseous media, Moscow, Nauka, 1991.

[6] Guckenheimer J. and Holmes P., Nonlinear oscillations, dynamical systems and bifurcations of vector fields, New York, Springer–Verlag, 1987.

[7] Vladimirov V.A., Periodic and soliton-like regimes in the gradient model of structured media, *Dopovidi NAS of Ukraine*, 2004, to appear.

[8] Vladimirov V.A., Modeling systems for relaxing media. Symmetry, restrictions and attractive features of invariant solutions, in Proceedings of Third International Conference "Symmetry in Nonlinear Mathematical Physics" (12–18 July, 1999, Kyiv), Editors A.G. Nikitin and V.M. Boyko, Kyiv, Insitute of Mathematics, 2000, V.30, Part 1, 232–238.