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# Symmetries and Supersymmetries of the Dirac-Type Operators on Curved Spaces

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The role of the Killing–Yano tensors in the construction of the Dirac-type operators is pointed out. The general results are applied to the case of the four-dimensional Euclidean Taub–Newman–Unti–Tamburino space. Three new Dirac-type operators, equivalent to the standard Dirac operator, are constructed from the covariantly constant Killing–Yano tensors of this space. Finally the Runge–Lenz operator for the Dirac equation in this background is expressed in terms of the fourth Killing–Yano tensor that is not covariantly constant.

## 1 Introduction

The (skew-symmetric) Killing–Yano (K-Y) tensors that were first introduced by Yano [1] from purely mathematical reasons, are profoundly connected to the supersymmetric classical and quantum mechanics on curved manifolds where such tensors do exist [2]. The K-Y tensors play an important role in theories with spin and especially in the Dirac theory on curved space-times where they produce first order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one $D$ [3]. Another virtue of the K-Y tensors is that they enter as square roots in the structure of several second rank Stäckel–Killing tensors that generate conserved quantities in classical mechanics or conserved operators which commute with $D$. The construction of Carter and McLenaghan depended upon the remarkable fact that the (symmetric) Stäckel–Killing tensor $K_{\mu\nu}$ involved in the constant of motion quadratic in the four-momentum $p_\mu$

$$Z = \frac{1}{2} K^{\mu\nu} p_\mu p_\nu \tag{1}$$

has a certain square root in terms of K-Y tensors $f_{\mu\nu}$:

$$K_{\mu\nu} = f_{\mu\lambda} f^{\lambda}_{\cdot\,\nu}{}^{\cdot}. \tag{2}$$

The K-Y tensor here is a 2-form $f_{\mu\nu} = -f_{\nu\mu}$ which satisfies the equation

$$f_{\mu\nu;\lambda} + f_{\mu\lambda;\nu} = 0.$$

These attributes of the K-Y tensors lead to an efficient mechanism of supersymmetry, especially when the Stäckel–Killing tensor $K_{\mu\nu}$ in equation (1) is proportional to the metric tensor $g_{\mu\nu}$ and the corresponding K-Y tensors in equation (2) are covariantly constant. Then each tensor of this type $f^i$ gives rise to a Dirac-type operator $D^i$ representing a supercharge of the superalgebra $\{D^i, D^j\} \propto D^2 \delta_{ij}$.

The general results are applied to the case of the four-dimensional Euclidean Taub–Newman–Unti–Tamburino (Taub–NUT) space. The Euclidean Taub-NUT metric is involved in many modern studies in physics. This metric might give rise to the gravitational analog of the Yang–Mills instanton [4]. The Kaluza–Klein monopole of Gross and Perry [5] and of Sorkin [6] was obtained by embedding the Taub–NUT gravitational instanton into five-dimensional Kaluza–Klein theory. On the other hand, in the long-distant limit, the relative motion of two monopoles is approximately described by the geodesics of this space [7].

The Euclidean Taub–NUT space which is a hyper-Kähler manifold possessing three covariantly constant K-Y tensors. By means of these covariantly constant Killing–Yano tensors it is possible to construct new Dirac-type operators [3] which anticommute with the standard Dirac operator.

We show that the representation of the whole theory can be changed using the $U(2)$ transformations among them the $SU(2)$ ones are generated just by the spin-like operators constructed using the above-mentioned three Killing–Yano tensors [8, 9]. On the basis of these results, we define the parity transformation and a discrete group with eight elements formed by the transformations that relate to each other the four Dirac operators and their parity is transformed as well. We show that this discrete group is a realization of the quaternion group which is isomorphic to the dicyclic group of order eight.

The Taub–NUT space also possesses a Killing–Yano tensor which is not covariantly constant. The corresponding non-standard operator, constructed with the general rule [3] anticommutes with the standard Dirac operator but is not equivalent to it [10]. This non-standard Dirac operator is connected to the hidden symmetries of the space allowing construction of a conserved vector operator analogous to the Runge–Lenz vector of the Kepler problem. We shall discuss the behavior of this operator under discrete transformations pointing out that the hidden symmetries are in some sense decoupled from the discrete symmetries studied here [10–15].

## 2 Dirac equation on a curved background

In what follows we shall consider the Dirac operator on a curved background that has the form

$$D_s = \gamma^\mu \hat{\nabla}_\mu . \tag{3}$$

In this expression the Dirac matrices $\gamma_\mu$ are defined in local coordinates by the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$ and $\hat{\nabla}_\mu$ denotes the canonical covariant derivative for spinors.

Carter and McLenaghan showed that in the theory of Dirac fermions for any isometry with Killing vector $R_\mu$ there is an appropriate operator [3]:

$$X_k = -i \left( R^\mu \hat{\nabla}_\mu - \frac{1}{4} \gamma^\mu \gamma^\nu R_{\mu;\nu} \right)$$

which commutes with the *standard* Dirac operator (3).

Moreover, each Killing–Yano tensor $f_{\mu\nu}$ produces a *non-standard* Dirac operator of the form

$$D_f = -i\gamma^\mu \left( f_\mu{}^\nu \hat{\nabla}_\nu - \frac{1}{6} \gamma^\nu \gamma^\rho f_{\mu\nu;\rho} \right)$$

which anticommutes with the standard Dirac operator $D_s$.

## 3 Dirac operators of the Taub–NUT space

Let us consider the Taub–NUT space and the chart with Cartesian coordinates $x^\mu$ $(\mu, \nu, \ldots = 1, 2, 3, 4)$ having the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{V} dl^2 + V(dx^4 + A_i dx^i)^2,$$

where $dl^2 = (d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ is the Euclidean three-dimensional line element and $\vec{A}$ is the gauge field of a monopole. Another chart suitable for applications is that of spherical coordinates $(r,\,\theta,\,\phi,\,\chi)$ among them the first three are the spherical coordinates commonly associated with the Cartesian space ones $x^i$ $(i, j, \ldots = 1, 2, 3)$, while $\chi + \phi = -x^4/\mu$. The real number $\mu$ is the parameter of the theory which enters in the form of the function $1/V(r) = 1 + \mu/r$. The only non-vanishing component of the vector potential in spherical coordinates is $A_\phi = \mu(1 - \cos\theta)$. This space has the isometry group $G_s = SO(3) \otimes U(1)_4$ formed by rotations of the Cartesian space coordinates and $x^4$ translations. The $U(1)_4$ symmetry is important since that eliminates so called NUT singularity if $x^4$ has the period $4\pi\mu$.

For the theory of the Dirac operators in Cartesian charts of the Taub–NUT space, it is convenient to consider the local frames given by tetrad fields $e(x)$ and $\hat{e}(x)$ as defined in [16] while the four Dirac matrices $\gamma^{\hat\alpha}$ that satisfy $\{\gamma^{\hat\alpha},\,\gamma^{\hat\beta}\} = 2\delta^{\hat\alpha\hat\beta}$ have to be written in the following representation

$$\gamma^i = -i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \tag{4}$$

where all of them are self-adjoint. In addition we consider the matrix

$$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}$$

which is denoted by $\gamma^0$ in Kaluza–Klein theory explicitly involving the time [12].

The *standard* Dirac operator of the theory without explicit mass term is defined as $D_s = \gamma^{\hat\alpha}\hat\nabla_{\hat\alpha}$ [12, 10] where the spin covariant derivatives with local indices $\hat\nabla_{\hat\alpha}$ depend on the momentum operators, $P_i = -i(\partial_i - A_i\partial_4)$ and $P_4 = -i\partial_4$, and spin connection [12], such that the Hamiltonian operator [12, 13]

$$H = \gamma^5 D_s = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$$

can be expressed in terms of Pauli operators,

$$\alpha = \sqrt{V}\left(\vec\sigma\cdot\vec{P} - \frac{iP_4}{V}\right), \qquad \alpha^* = V\left(\vec\sigma\cdot\vec{P} + \frac{iP_4}{V}\right)\frac{1}{\sqrt{V}},$$

involving the Pauli matrices $\sigma_i$. These operators give the (scalar) Klein–Gordon operator of the Taub–NUT space [12, 13], $\Delta = -\nabla_\mu g^{\mu\nu}\nabla_\nu = \alpha^*\alpha$. We specify that here the star superscript is a mere notation that does not represent the Hermitian conjugation, because we are using a non-unitary representation of the algebra of Dirac operators. Of course, this is *equivalent* to the unitary representation where all of these operators are self-adjoint [12].

The first three Killing–Yano tensors of the Taub–NUT space [17],

$$f^i = f^i_{\hat\alpha\hat\beta}\hat{e}^{\hat\alpha}\wedge\hat{e}^{\hat\beta} = 2\hat{e}^4\wedge\hat{e}^i + \varepsilon_{ijk}\hat{e}^j\wedge\hat{e}^k$$

are rather special since they are covariantly constant. $f^i$ define three anticommuting complex structures of the Taub–NUT manifold, their components realizing the quaternion algebra

$$f^i f^j + f^j f^i = -2\delta_{ij}, \qquad f^i f^j - f^j f^i = -2\varepsilon_{ijk}f^k.$$

Existence of these Killing–Yano tensors is linked to the hyper-Kähler geometry of the manifold and shows directly the relation between the geometry and the $N = 4$ supersymmetric extension of the theory [2, 18]. Moreover, we can give a *physical* interpretation of these Killing–Yano tensors defining the *spin-like* operators,

$$\Sigma_i = -\frac{i}{4} f^i_{\hat\alpha\hat\beta}\gamma^{\hat\alpha}\gamma^{\hat\beta} = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

that have similar properties to those of the Pauli matrices. In the pseudo-classical description of a Dirac particle [2, 18] the covariantly constant Killing–Yano tensors correspond to components of the spin which are separately conserved.

Here, since the Pauli matrices commute with the Klein–Gordon operator, the spin-like operators (5) commute with $H^2$. Remarkable existence of the Killing–Yano tensors allows one to construct *Dirac-type* operators [3]

$$Q_i = -i f^i_{\hat\alpha\hat\beta}\gamma^{\hat\alpha}\hat\nabla^{\hat\beta} = \{H, \Sigma_i\} = \begin{pmatrix} 0 & \sigma_i\alpha^* \\ \alpha\sigma_i & 0 \end{pmatrix} \tag{6}$$

that anticommute with $D_s$ and $\gamma^5$ and commute with $H$ [10]. Another Dirac operator can be defined using the fourth Killing–Yano tensor but this will be discussed separately in Section 6.

## 4 Equivalent representations

In [12] we have shown that in the massless case the operators $Q_i$ ($i = 1, 2, 3$) and the new supercharge $Q_0 = iD_s = i\gamma^5 H$ form the basis of a $N = 4$ superalgebra obeying the anticommutation relations

$$\{Q_A, Q_B\} = 2\delta_{AB}H^2, \qquad A, B, \ldots = 0, 1, 2, 3 \tag{7}$$

linked to the hyper-Kähler geometric structure of the Taub–NUT space. In addition, we associate to each Dirac operator $Q_A$ its own Hamiltonian operator $\tilde{Q}_A = -i\gamma^5 Q_A$ obtaining thus another set of supercharges,

$$\tilde{Q}_0 = H, \qquad \tilde{Q}_i = i[H, \Sigma_i] \tag{8}$$

that obey the same anticommutation relations as (7). Thus we find that there are two similar superalgebras of operators with precise physical meaning. Obviously, since all of these operators must be self-adjoint we have to work only with *unitary* representations of these superalgebras, up to an equivalence.

The concrete form of these supercharges depends on the representation of the Dirac matrices which can be changed at any time with the help of a non-singular operator $T$ such that all of the $4 \times 4$ matrix operators of the Dirac theory transform as $X \to X' = TXT^{-1}$. In this way one obtains an *equivalent* representation which preserves the commutation and the anticommutation relations. In [12] we have used such transformations for pointing out that the convenient representations where we work are equivalent to unitary one. We note that some properties of the transformations changing representations in theories with two Dirac operators and their possible new applications are discussed in [19].

For example, simple and convenient transformations of the form can be chosen:

$$U(\beta, \vec\xi) = \begin{pmatrix} \hat{U}(\beta, \vec\xi) & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix},$$

where $\hat{U}(\beta, \vec\xi) = e^{-i\beta}\hat{U}(\vec\xi) \in U(2) = U(1) \otimes SU(2)$ with $\hat{U}(\vec\xi) \in SU(2)$. This is because among these transformations one could find those linking equivalent Dirac operators.

It is interesting to observe that the $SU(2)$ transformations are generated just by the above defined spin-like operators as

$$U(\vec{\xi}) = U(0, \vec{\xi}) = e^{-i\vec{\xi}\cdot\vec{\Sigma}/2} = \begin{pmatrix} \hat{U}(\vec{\xi}) & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}. \tag{9}$$

If we take now $\vec{\xi} = 2\varphi\,\vec{n}$ with $|\vec{n}| = 1$ and $\varphi \in [0, \pi]$, we find that

$$\hat{U}(\vec{\xi}) = e^{-i\vec{\xi}\cdot\vec{\sigma}/2} = \mathbf{1}_2 \cos\varphi - i\vec{n}\cdot\vec{\sigma}\sin\varphi$$

and after a little calculation we can write the concrete action of (9) as

$$Q'_0 = U(\vec{\xi})Q_0U^+(\vec{\xi}) = Q_0\cos\varphi + n_iQ_i\sin\varphi, \tag{10}$$

$$Q'_i = U(\vec{\xi})Q_iU^+(\vec{\xi}) = Q_i\cos\varphi - (n_iQ_0 + \varepsilon_{ijk}n_jQ_k)\sin\varphi. \tag{11}$$

Hereby we see that the supercharges are mixed among themselves in linear combinations involving only *real* coefficients. In addition, we observe that these transformations correspond to an irreducible representation since the supercharges transform like the real components of a Pauli spinor. In other words, the usual $SU(2)$ transformations $\psi_Q \to \psi'_Q = \hat{U}^+(\vec{\xi})\psi_Q$ of the spinor-operator

$$\psi_Q = \begin{pmatrix} Q_0 - iQ_3 \\ Q_2 - iQ_1 \end{pmatrix}$$

give just the transformations (10) and (11).

## 5 Discrete transformations

Let us focus now only on the transformations which transform the supercharges $Q_A$ among themselves without affecting their form. From (10) and (11) we see that there exist particular transformations,

$$Q_k = U_kQ_0U_k^+, \qquad k = 1, 2, 3,$$

where the matrix $U_k = \mathrm{diag}(-i\sigma_k, \mathbf{1}_2)$ is given by $-i\sigma_k \in SU(2)$. In addition, we consider the *parity* operator $P = P^{-1} = -\gamma^5$ that changes the sign of supercharges,

$$PQ_AP = -Q_A, \qquad A = 0, 1, 2, 3. \tag{12}$$

Then it is not hard to verify that the identity $I = \mathbf{1}_4$, $P$ and the sets of matrices $U_k$ and $PU_k$ ($k = 1, 2, 3$) form a discrete group of order eight, the multiplication table of which is determined by the following rules

$$\begin{aligned} &P^2 = I, \qquad PU_k = U_kP, \\ &{U_1}^2 = {U_2}^2 = {U_3}^2 = P, \\ &U_1U_2 = U_3, \qquad U_2U_1 = PU_3, \quad \dots \quad \text{etc.} \end{aligned} \tag{13}$$

We denote this group by $\mathcal{G}_Q$ since it is a realization of the quaternion group $\mathbf{Q}$ which is isomorphic with the dicyclic group $\langle 2, 2, 2\rangle$ [20,21]. In the representation (4) of the $\gamma$-matrices, its operators are defined by proper unitary matrices (which satisfy $G^{-1} = G^+$ and $\det G = 1$, $\forall\, G \in \mathcal{G}_Q$) constructed using the elements $\pm\mathbf{1}_2, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3$ of the natural realization of $\mathbf{Q}$ as a discrete subgroup of $SU(2)$.

The group $\mathcal{G}_Q$ is interesting because it brings together the parity that produces the transformations (12) and the operators $U_k$ giving sequences of the form

$$Q_1 = U_3^+ Q_2 U_3 = U_2 Q_3 U_2^+ = U_1 Q_0 U_1^+, \quad \ldots \quad \text{etc.}$$

that leads to the conclusion that the Dirac operators and their parity transformed $\pm Q_A$ $(A = 0,1,2,3)$, are equivalent among themselves. All these operators constitute the orbit $\Omega_Q = \{Q \,|\, Q = G Q_0 G^+, \ \forall G \in \mathcal{G}_Q\}$ of the group $\mathcal{G}_Q$ in the algebra of the $4 \times 4$ matrix operators. A similar orbit $\tilde{\Omega}_Q$ can be constructed for the associated Hamiltonian operators $\pm\tilde{Q}_A$ defined by (8), if we start with $\tilde{Q}_0$ instead of $Q_0$. It is remarkable that each of these two orbits includes *only* operators representing (up to sign) supercharges obeying superalgebras of the form (7).

In the Kaluza–Klein theory with the time trivially added [12], the time dependent term of the whole massless Dirac operator commutes with all the operators of $\mathcal{G}_Q$ so as it remains unchanged when one replaces the space parts using the discrete transformations of this group. In these conditions all the Dirac operators of $\Omega_Q$ lead to equivalent Dirac equations from the physical point of view. These can be written in Hamiltonian form as $i\partial_t \psi_A^{(\pm)} = \pm\tilde{Q}_A \psi_A^{(\pm)}$ $(A = 0,1,2,3)$ and produce the same energy spectrum which coincides with that of the Klein–Gordon equation as it results from the superalgebra (7) [22, 12].

The existence of this discrete symmetry among the four supercharges of the superalgebra of the Dirac and Dirac-type operators (or the corresponding Hamiltonian operators) must be understood as a consequence of the fact that the Taub-NUT space has a hyper-Kähler structure modeled on a quaternion inner-product space [23]. In other words, the Dirac theory in this space picks up the basic quaternion character of the tangent space showing it off as the discrete symmetry due to the group $\mathcal{G}_Q \sim \mathbf{Q}$, naturally related to the specific supersymmetries of this geometry.

## 6 Hidden symmetries and the fifth Dirac operator

In the Taub–NUT space, in addition to the above discussed covariantly constant Killing–Yano tensors, there exists a fourth Killing–Yano tensor,

$$f^Y = -\frac{x^i}{r} f^i + \frac{2x^i}{\mu V} \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k,$$

which is not covariantly constant. The presence of $f^Y$ is due to existence of the hidden symmetries of the Taub–NUT geometry that are encapsulated in three non-trivial Stäckel–Killing tensors. These are interpreted as the components of the so-called Runge–Lenz vector of the Taub–NUT problem and are expressed as symmetrized products of the Killing–Yano tensors $f^Y$ and $f^i$ $(i = 1,2,3)$ [24].

As in the case of the Dirac operators (6), one can use $f^Y$ for defining the fifth Dirac operator

$$Q_0^Y = -i\gamma^{\hat{\alpha}} \left( f^Y_{\hat{\alpha}\hat{\beta}} \hat{\nabla}^{\hat{\beta}} - \frac{1}{6} \gamma^{\hat{\beta}} \gamma^{\hat{\delta}} f^Y_{\hat{\alpha}\hat{\beta};\hat{\delta}} \right),$$

called here the *non-standard* or *hidden* Dirac operator to emphasize the connection with the hidden symmetry of the Taub–NUT problem. It is denoted by $Q_0^Y$ instead of $Q^Y$ as in [10] to point out its relation to the standard Dirac operator since it can be put in the form

$$Q_0^Y = i\frac{r}{\mu} \left[ Q_0, \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_r V^{-1} \end{pmatrix} \right],$$

where $\sigma_r = \vec{x}\cdot\vec{\sigma}/r$. We showed that $Q_0^Y$ commutes with $\tilde{Q}_0 = H$ and anticommutes with $Q_0$ and $\gamma^5$ [10]. This operator is important because it allowed us to derive the explicit form of the Runge–Lenz operator $\vec{K}$ of the Dirac field in Taub–NUT background establishing its properties [10].

We recall that the components of the conserved total angular momentum $\vec{J}$, and the operators $R_i = F^{-1}K_i$ with $F^2 = {P_4}^2 - H^2$ are just the generators of the dynamical algebra of the Dirac theory in Taub–NUT background [13, 14].

Starting with $Q_0^Y$ we can construct a new orbit $\Omega^Y$ of $\mathcal{G}_Q$ defining

$$Q_k^Y = U_k Q_0^Y U_k^+ = i\frac{r}{\mu}\left[Q_k, \begin{pmatrix} \sigma_k\sigma_r\sigma_k & 0 \\ 0 & \sigma_r V^{-1} \end{pmatrix}\right] \tag{14}$$

(for $k = 1, 2, 3$) and observing that $PQ_A^Y P = -Q_A^Y$, $A = 0, 1, 2, 3$.

From the explicit form (14) we deduce that, in contrast with the operators of the orbits $\Omega_Q$ and $\tilde{\Omega}_Q$, those of the orbit $\Omega^Y$ have more involved algebraic properties. We can show that calculating

$$H^2(Q_0^Y)^2 = H^4 + \frac{4}{\mu^2}H^2\left(\vec{J}^2 + \frac{1}{4}\right) + 4F^2P_4^2, \tag{15}$$

and it is worth comparing it with equation (7). The Dirac-type operators $Q_A$ are characterized by the fact that their quantal anticommutator closes on the square of the Hamiltonian of the theory. No such expectation applies to the non-standard, hidden Dirac operators $Q_A^Y$ which close on a combination of different conserved operators. Also from equation (15) it results that $(Q_A^Y)^2 \neq (Q_B^Y)^2$ if $A \neq B$ (because $\vec{J}^2$ does not commute with $U_k$). Moreover, one can show that the commutators $[Q_A^Y, Q_B^Y]$ have complicated forms that can not be expressed in terms of operators $Q_A^Y$. Therefore, neither the commutator nor the anticommutator of the pairs of operators of this orbit do not lead to significant algebraic results as the anticommutation relations (7) of the operators $Q_A$ $(A = 0, 1, 2, 3)$.

## 7 Concluding remarks

In the study of the Dirac equation in curved spaces, it has been proved that the Killing–Yano tensors play an essential role in the construction of new Dirac-type operators. The Dirac-type operators constructed with the aid of covariantly constant Killing–Yano tensors are equivalent with the standard Dirac operator. The non-covariantly constant Killing–Yano tensors generate non-standard Dirac operators that are not equivalent to the standard Dirac operator and they are associated with the hidden symmetries of the space.

The Taub–NUT space has a special geometry where the covariantly constant Killing–Yano tensors exist by virtue of the metric being self-dual and the Dirac-type operators generated by them are equivalent to the standard one. The fourth Killing–Yano tensor $f^Y$ that is not covariantly constant exists by virtue of the metric being of type $D$. The corresponding non-standard or hidden Dirac operator does not close on $H$ as it can be seen from equation (15) and is not equivalent to the Dirac-type operators. As it was mentioned, it is associated with the hidden symmetries of the space allowing the construction of the conserved vector-operator analogous to the Runge–Lenz vector of the Kepler problem.

Let us mention that in the pseudo-classical spinning particle models in curved spaces from covariantly constant K-Y tensors $f_{\mu\nu}$ conserved quantities of the type $f_{\mu\nu}\theta^\mu\theta^\nu$ depending on the Grassmann variables $\{\theta^\mu\}$ can be constructed [24]. The Grassmann variables $\{\theta^\mu\}$ transform as a tangent space vector and describe the spin of the particle. The antisymmetric tensor $S^{\mu\nu} = -i\theta^\mu\theta^\nu$ generates the internal part of the local tangent space rotations. For example, in the spinning Euclidean Taub–NUT space such operators correspond to components of the spin that are separately conserved [18].

The construction of the new supersymmetries in the context of pseudo-classical mechanics can be carried over straightforwardly to the case of quantum mechanics by the usual replacement of phase space coordinates by operators and Poisson–Dirac brackets by anticommutators [25]. In terms of these operators the supercharges are replaced by Dirac-type operators [26]. In both cases, the correspondence principle leads to equivalent algebraic structures making obvious the relations between these approaches [18].

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