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# Construction of Special Solutions for Nonintegrable Dynamical Systems with the Help of the Painlevé Analysis

*Sergey Yu. VERNOV*

*Skobeltsyn Institute of Nuclear Physics, Moscow State University, Vorob'evy Gory, 119992 Moscow, Russia*
E-mail: *svernov@theory.sinp.msu.ru*

The generalized Hénon–Heiles system has been considered. In two nonintegrable cases with the help of the Painlevé test new special solutions have been found as Laurent series, depending on three parameters. The obtained series converge in some ring. One of parameters determines the singularity point location, other parameters determine coefficients of series. For some values of these parameters the obtained Laurent series coincide with the Laurent series of the known exact solutions. The Painlevé test can be used not only to construct local solutions as the Laurent series but also to find elliptic solutions.

## 1 The Painlevé property and integrability

A Hamiltonian system in a $2s$-dimensional phase space is called *completely integrable* (Liouville integrable) if it possesses $s$ independent integrals that commute with respect to the associated Poisson bracket. When this is the case, the equations of motion are (in principal, at least) separable and solutions can be obtained by the method of quadratures.

When some mechanical problem is studied, time is assumed to be real, whereas the integrability of motion equations depends on the behavior of their solutions as functions of complex time. S.V. Kovalevskaya was the first who proposed [1] to interpret time as a complex variable and to require that solutions of mechanical problems have to be single-valued functions meromorphic in the entire complex plane. This idea gave a remarkable result: S.V. Kovalevskaya discovered a new integrable case (nowadays known as the Kovalevskaya's case) for the motion of a heavy rigid body about a fixed point [1] (see also [2, 3]). The Kovalevskaya's result demonstrated that the analytic theory of differential equations can be fruitfully applied to mechanical and physical problems. The important stage of development of this theory was the Painlevé classification of ordinary differential equations (ODEs) with respect to the types of singularities of their solutions [4].

Let us formulate the Painlevé property for ODEs. Solutions of a system of ODEs regarded as analytic functions may have isolated singularity points [5, 6]. A singularity point of a solution is said to be *critical* (as opposed to *noncritical*) if the solution is multivalued (single-valued) in its neighborhood and *movable* if its location depends on initial conditions. The *general solution* of an ODE of order $N$ is the set of all solutions mentioned in the existence theorem of Cauchy, i.e. determined by the initial values. It depends on $N$ arbitrary independent constants. A *special solution* is any solution obtained from the general solution by giving values to the arbitrary constants. A *singular solution* is any solution which is not special, i.e. which does not belong to the general solution.

**Definition 1.** A system of ODEs has ***the Painlevé property*** if its general solution has no movable critical singularity point [4].

Investigations of many dynamical systems [7] show that systems with the Painlevé property are completely integrable. Arguments, which clarify the connection between the Painlevé analysis and the existence of motion integrals, are presented in [8, 9]. At the same time the integrability of an arbitrary system with the Painlevé property has yet to be proved. There is no algorithm for construction of additional integrals by the Painlevé analysis. It is easy to give an example of an integrable system without the Painlevé property [10]. The system with the Hamiltonian $H = \frac{1}{2}p^2 + f(x)$, where $f(x)$ is a polynomial whose power is not lower than five, is trivially integrable, but its general solution is not a meromorphic function.

The study of complex-time singularities is a useful tool for analysis of not only integrable systems, but also of chaotic dynamics [11]. The Painlevé analysis can be connected to the normal form theory [12].

The ***Painlevé test*** is any algorithm that checks some necessary conditions for a differential equation to have the Painlevé property. The original algorithm, developed by P. Painlevé and used by him to find all the second order ODEs with the Painlevé property, is known as the $\alpha$-method. The method of S.V. Kovalevskaya is not as general as the $\alpha$-method, but much simpler. The remarkable property of this test is that it can be checked in a finite number of steps. This test can only detect the occurrence of logarithmic and algebraic branch points. Up to the present there is no general finite algorithmic method to detect the occurrence of essential singularities. Different variants of the Painlevé test are compared in [13, R. Conte paper].

Developing the Kovalevskaya method [1] further, M.J. Ablowitz, A. Ramani and H. Segur constructed a new algorithm of the Painlevé test for ODEs [14]. They also were the first to point out the connection between the nonlinear partial differential equations (PDEs) that are solvable by the inverse scattering transform method, and ODEs with the Painlevé property. Subsequently the Painlevé property for PDEs was defined and the corresponding Painlevé test (the WTC procedure) was constructed [15, 16] (see also [13, 17]). With the help of this test it was found that all PDEs solvable by inverse scattering transforms, have the Painlevé property, maybe after some change of variables. For many integrable PDEs, for example, the Korteweg–de Vries equation [7], the Bäcklund transformations and the Lax representations result from the WTC procedure [16, 18]. For certain nonintegrable PDEs special solutions were constructed using this algorithm [19, 20].

The algorithm for finding of special solutions for ODEs into the form of a finite expansion in powers of unknown function $\varphi(t - t_0)$ was constructed in [21, 22]. The function $\varphi(t - t_0)$ and coefficients have to satisfy some system of ODE, often simpler than an initial one. This method has been used to construct exact special solutions for some nonintegrable ODEs [23, 24]. With the help of the perturbative Painlevé test [17] a four-parameter generalization of an exact three-parameter solution of the Bianchi IX cosmological model was constructed [25].

## 2 The Hénon–Heiles Hamiltonian

In the 1960s the models of the star motion in an axial-symmetric and time-independent potentials were developed to show either existence or absence of the third integral for some polynomial potentials. Due to the symmetry of the potential the considered system is equivalent to two-dimensional one. To clarify the question of the existence of the third integral Hénon and Heiles [26] considered behavior of numerically integrated trajectories. Emphasizing that their choice does not proceed from experimental data, they have proposed the Hamiltonian

$$H = \frac{1}{2}\left(x_t^2 + y_t^2 + x^2 + y^2\right) + x^2 y - \frac{1}{3}y^3,$$

because on the one hand, it is analytically simple; this makes the numerical computations of trajectories easy; on the other hand, it is sufficiently complicated to give trajectories that are far from trivial. Indeed, for low energies the Hénon–Heiles system appears to be integrable, in so much as trajectories (numerically integrated) always lay on well-defined two-dimensional surfaces. On the other hand, for high energies many of these integral surfaces are destroyed, it points to absence of the third integral.

The generalized Hénon–Heiles system is described by the Hamiltonian:

$$H = \frac{1}{2}\left(x_t^2 + y_t^2 + \lambda x^2 + y^2\right) + x^2 y - \frac{C}{3}\,y^3 \tag{1}$$

and the corresponding system of the motion equations:

$$\begin{aligned} x_{tt} &= -\lambda x - 2xy,\\ y_{tt} &= -y - x^2 + Cy^2, \end{aligned} \tag{2}$$

where $x_{tt} \equiv \frac{d^2x}{dt^2}$ and $y_{tt} \equiv \frac{d^2y}{dt^2}$, $\lambda$ and $C$ are numerical parameters.

The generalized Hénon–Heiles system is a model, not only actively investigated by various mathematical methods (see [27] and references therein), but also widely used in physics, in particular, in gravitation [28, 29]. The models, described by the Hamiltonian (1) with some additional nonpolynomial terms, are actively studied as well [30–32].

Due to the Painlevé analysis the following integrable cases of (2) were found:

(i) $C = -1$, $\lambda = 1$,
(ii) $C = -6$, $\lambda$ is an arbitrary number,
(iii) $C = -16$, $\lambda = 1/16$.

The general solutions in the analytic form are known only in integrable cases [31, 32], in other cases not only four-, but even three-parameter exact solutions have yet to be found. In nonintegrable cases local four-parameter solutions as converging psi-series solutions were found [33] for all values of the parameter $C$, except $C = -2$. The Ablowitz–Ramani–Segur algorithm of the Painlevé test appears very useful to find such values of parameter at which three-parameter solutions can be expanded into formal Laurent series and to construct these local solutions. The knowledge of local solutions can be used to find solutions in the analytical form.

Let us assume that the behavior of solutions in a sufficiently small neighborhood of the singularity point is algebraic, i.e., $x$ and $y$ tend to infinity as

$$x = a_\alpha (t - t_0)^\alpha \qquad \text{and} \qquad y = b_\beta (t - t_0)^\beta,$$

where $\alpha$, $\beta$, $a_\alpha$ and $b_\beta$ are some constants.

The Painlevé test gives all information about behavior of solutions in the neighborhood of the singularity point (see, for example, [7]). There exist two possible variants of dominant behavior and resonance structure of solutions of the generalized Hénon–Heiles system [7, 33]:

| *Case 1*: | *Case 2*: ($\beta < \Re e(\alpha)$) |
|---|---|
| $\alpha = -2$, | $\alpha = \frac{1 \pm \sqrt{1-48/C}}{2}$, |
| $\beta = -2$, | $\beta = -2$, |
| $a_\alpha = \pm 3\sqrt{2+C}$, | $a_\alpha = c_1$ (an arbitrary number), |
| $b_\beta = -3$, | $b_\beta = \frac{6}{C}$, |
| $r = -1,\ 6,\ \frac{5}{2} - \frac{\sqrt{1-24(1+C)}}{2},\ \frac{5}{2} + \frac{\sqrt{1-24(1+C)}}{2}$ | $r = -1,\ 0,\ 6,\ \mp\sqrt{1 - \frac{48}{C}}$ |

The values of $r$ denote resonances: $r = -1$ corresponds to arbitrary parameter $t_0$; $r = 0$ (in the *Case 2*) corresponds to arbitrary parameter $c_1$. Other values of $r$ determine powers of $t$, to be exact, $t^{\alpha+r}$ for $x$ and $t^{\beta+r}$ for $y$, at which new arbitrary parameters can appear.

For integrability of system (2) all values of $r$ have to be integer and all systems with zero determinants have to have solutions at any values of free parameters. It is possible only in the integrable cases (i)–(iii).

Those values of $C$, at which $r$ are integer numbers either only in the *Case 1* or only in the *Case 2*, are of interest for search of special three-parameter solutions. Those cases where an additional negative resonance is present likely correspond to singular, rather the general, solutions [7].

Let us consider all cases, when there exist special (not singular) solutions, representable as three-parameter Laurent series (maybe, multiplied on $\sqrt{t-t_0}$). From the requirement that all values of $r$ but one are integer and nonnegative numbers, we obtain the following values of $C$: $C=-1$, $C=-4/3$ (the *Case 1*), $C=-16/5$, $C=-6$, $C=-16$ (the *Case 2,* $\alpha=\frac{1-\sqrt{1-48/C}}{2}$) and $C=-2$, when two types of singular behavior coincide.

At $C=-2$ (in the *Case 1*) $a_\alpha=0$. This is the consequence of the fact that, contrary to our assumption, the behavior of the solution in the neighborhood of a singularity point is not algebraic, because its dominant term includes logarithm [7]. At $C=-6$ and any value of $\lambda$ exact four-parameter solutions are known. In cases $C=-1$ and $C=-16$ substitution of unknown function as Laurent series gives the equations in $\lambda$: accordingly $\lambda=1$ and $\lambda=1/16$; hence, in nonintegrable cases special three-parameter local solutions have to include logarithmic terms. Single-valued three-parameter special solutions can exist only in two nonintegrable cases, at $C=-16/5$ and at $C=-4/3$.

## 3 New solutions

Let us consider the Hénon–Heiles system with $C=-16/5$. In the *Case 1* some values of $r$ are not rational. To find special three-parameter solutions we consider the *Case 2*. In this case $\alpha=-3/2$ and $r=-1,\ 0,\ 4,\ 6$, hence, in the neighborhood of the singularity point $t_0$ we have to seek $x$ in such a form that $x^2$ can be expanded into Laurent series, beginning with $(t-t_0)^{-3}$. Let $t_0=0$, substituting

$$x=\sqrt{t}\left(c_1t^{-2}+\sum_{j=-1}^{\infty}a_jt^j\right) \qquad \text{and} \qquad y=-\frac{15}{8}t^{-2}+\sum_{j=-1}^{\infty}b_jt^j \tag{3}$$

in (2), we obtain the following sequence of linear system in $a_k$ and $b_k$:

$$\begin{aligned}
&\left(k^2-4\right)a_k+2c_1b_k=-\lambda a_{k-2}-2\sum_{j=-1}^{k-1}a_jb_{k-j-2},\\
&((k-1)k-12)b_k=-b_{k-2}-\sum_{j=-2}^{k-1}a_ja_{k-j-3}-\frac{16}{5}\sum_{j=-1}^{k-1}b_jb_{k-j-2}.
\end{aligned} \tag{4}$$

The determinants of the systems (4) corresponding to $k=2$ and $k=4$ are equal to zero. To determine $a_2$ and $b_2$ we have the following system:

$$\begin{aligned}
&c_1\big(557056c_1^8+(15552000\lambda-4860000)c_1^4+864000000b_2\\
&\qquad+108000000\lambda^2-67500000\lambda+10546875\big)=0,\\
&818176c_1^8+(15660000\lambda-4893750)c_1^4-810000000b_2-6328125=0.
\end{aligned} \tag{5}$$

It is easy to see that this system contains no terms proportional to $a_2$, therefore, $a_2$ is the new constant of integration. We discard the solution with $c_1=0$ and obtain the system in $c_1^4$

and $b_2$. System (5) has solutions only if

$$c_1^4 = \frac{1125\left(525 - 1680\lambda \pm 4\sqrt{35(2048\lambda^2 - 1280\lambda + 387)}\right)}{167552}.$$

We obtain new constant of integration $a_2$, but we must fix $c_1$, so number of constants of integration is equal to 2. It is easy to verify that $b_4$ is an arbitrary parameter, because the corresponding system is equivalent to one linear equation. System (2) is invariant under exchange $x$ to $-x$, so we obtain four different local solutions which depend on three parameters, namely $t_0$, $a_2$ and $b_4$. With the help of some computer algebra system, for example, **REDUCE** [34], these solutions can be obtained with arbitrary accuracy. For the case $\lambda = 1/9$ the obtained Laurent series are presented in [35].

At $C = -4/3$ the situation is similar. In the *Case 1* we have $r = -1,\ 1,\ 4,\ 6$. Substituting

$$x = \sqrt{6}t^{-2} + \sum_{k=-1}^{\infty} d_k t^k \qquad \text{and} \qquad y = -3t^{-2} + \sum_{k=-1}^{\infty} f_k t^k \tag{6}$$

in system (2), we receive a sequence of linear systems in $d_k$ and $f_k$:

$$\begin{aligned} &((k-1)k-6)d_k + 2\sqrt{6}f_k = -\lambda d_{k-2} - 2\sum_{j=-1}^{k-1} d_j f_{k-j-2}, \\ &2\sqrt{6}d_k + ((k-1)k-8)f_k = -f_{k-2} - \sum_{j=-1}^{k-1} d_j d_{k-j-2} - \frac{4}{3}\sum_{j=-1}^{k-1} f_j f_{k-j-2}. \end{aligned} \tag{7}$$

The determinants of the systems (7) corresponding to $k = -1,\ 2,\ 4$ are equal to zero. The first system ($k = -1$) always has infinite number of solutions and $f_{-1}$ is a parameter. We have to fix this parameter to solve the system corresponding to $k = 2$. This system has solutions only if

$$f_{-1}^2 = \frac{105 - 140\lambda \pm \sqrt{7\left(1216\lambda^2 - 1824\lambda + 783\right)}}{385} \qquad \text{or} \qquad f_{-1} = 0.$$

At $k = 4$ system (7) is reduced to one equation. Thus, at $C = -4/3$ we have five three-parameter ($t_0$, $f_2$ and $f_4$) solutions.

The convergence of all the Laurent series solutions on some real time interval have been proved in [33]. For the obtained solutions it is easy to find conditions at which the series converge at $0 < |t| \leqslant 1 - \varepsilon$, where $\varepsilon$ is any positive number. Our series converge in the above-mentioned ring, if $\exists\, N$ and $\exists\, M$ such that $\forall\, n > N \ \ |a_n| \leqslant M$ and $|b_n| \leqslant M$. Let $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for all $-1 < n < k$, then (in the case $C = -16/5$) from (4) we obtain:

$$|a_k| \leqslant \frac{2M(k+1) + |\lambda| + 2|c_1|}{|k^2 - 4|}M, \qquad |b_k| \leqslant \frac{21Mk + 26M + 5}{5|k^2 - k - 12|}M.$$

It is easy to see that there exists such $N$ that if $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for $-1 \leqslant n \leqslant N$, then $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for $-1 \leqslant n < \infty$. So one can prove the convergence, analyzing values of a finite number of the first coefficients of series. For $C = -4/3$ it is easy to obtain the analogous result.

## 4 Global single-valued solutions

We have found some local three-parameter solutions. To seek for the global single-valued solutions we transform system (2) into the fourth order equation [36, 37]:

$$y_{tttt} = (2C-8)y_{tt}y - (4\lambda+1)y_{tt} + 2(C+1)y_t^2 + \frac{20C}{3}y^3 + (4C\lambda - 6)y^2 - 4\lambda y - 4H. \tag{8}$$

Point out that the energy of the system $H$ and initial data $y_0$, $y_{0t}$, $y_{0tt}$ and $y_{0ttt}$ are dependent. If $y_{0tt} = Cy_0^2 - y_0$, then there exists the connection $y_{0ttt} = 2Cy_0y_{0t} - y_{0t}$, in opposite case $H$ is a function of initial data.

There are some reasons to seek three-parameter solutions of equation (8) in terms of elliptic functions. In 1999 E.I. Timoshkova [37] found that the general solution of the following equation:

$$y_t^2 = \tilde{\mathcal{A}}y^3 + \tilde{\mathcal{B}}y^2 + \tilde{\mathcal{C}}y + \tilde{\mathcal{D}} + \tilde{\mathcal{G}}y^{5/2} + \tilde{\mathcal{E}}y^{3/2}$$

with some values of constants $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{D}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{E}}$, is one-parameter solution of equation (8) in each of two above-mentioned nonintegrable cases ($C = -4/3$ or $C = -16/5$, $\lambda$ is an arbitrary number). If $\tilde{\mathcal{G}} = 0$ and $\tilde{\mathcal{E}} = 0$ we obtain the well-known solutions in terms of the Weierstrass elliptic function. Solutions with $\tilde{\mathcal{G}} \neq 0$ or $\tilde{\mathcal{E}} \neq 0$ are derived only at $\tilde{\mathcal{D}} = 0$, therefore, substitution $y(t) = \varrho(t)^2$ gives:

$$\varrho_t^2 = \frac{1}{4}\big(\tilde{\mathcal{A}}\varrho^4 + \tilde{\mathcal{G}}\varrho^3 + \tilde{\mathcal{B}}\varrho^2 + \tilde{\mathcal{E}}\varrho + \tilde{\mathcal{C}}\big). \tag{9}$$

Two-parameter solutions $y(t) = \varrho(t)^2 + P_0$, where $P_0$ is an arbitrary constant and $\varrho(t)$ satisfies equation (9), were obtained in [38, 39]. These solutions are the following elliptic functions

$$y(t-t_0) = \left(\frac{a\wp(t-t_0)+b}{c\wp(t-t_0)+d}\right)^2 + P_0, \qquad ad - bc = 1, \tag{10}$$

where $\wp(t - t_0)$ is the Weierstrass elliptic function, $a$, $b$, $c$ and $d$ are some constants. The parameter $P_0$ defines the energy of the system. There exist two different elliptic solutions for each possible pair of values of $C$ and $\lambda$.

Let us consider the three-parameter solutions that were obtained at $C = -4/3$. If we choose $f_{-1} = 0$, then we obtain the solution which generalizes the well-known two-parameter solution in terms of Weierstrass elliptic functions. Other solutions generalize two-parameter solutions, obtained in [39]. The coefficient $f_{-1}$ is a residue of $y$. The sum of residues of an elliptic function in its parallelogram of periods has to be zero [40], hence, two local solutions with opposite signs of $f_{-1}$ correspond to one global elliptic solution. The obtained local three-parameter solution generalizes the Laurent series of the two-parameter elliptic solutions in the form (10).

For $C = -16/5$ we obtained four local solutions, which generalize two global elliptic solutions in the form (10). So, each obtained local three-parameter solution generalize the Laurent series of some two-parameter elliptic solution, and we can assume that unknown global three-parameter solutions are elliptic functions.

Of course, solutions, which are single-valued in the neighborhood of one singularity point, can be multivalued in the neighborhood of another singularity point. So, we can only assume that global three-parameter solutions are single-valued. If we assume this and, moreover, that these solutions are elliptic functions (or some degenerations of them), then we can seek them as solutions of some polynomial first order equations. The classical theorem, which was established by Briot and Bouquet [41], proves that if the general solution of the autonomous polynomial first order ODE is single-valued, then this solution is either an elliptic function, or a rational function of $e^{\gamma x}$, $\gamma$ being some constant, or a rational function of $x$. Note that the third case is a degeneracy of the second one that in its turn is a degeneracy of the first one. At the same time, there exist elementary functions, for example, the function $f(t) = t + \sin(t)$ that are not solutions of any first order polynomial ODE.

A second result, of immediate practical use, is due to P. Painlevé [4]. He has proved that if the general solution of the autonomous polynomial first order ODE is single-valued, then the necessary form of this ODE is

$$F(y, y_t) \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} h_{jk} y^j {y_t}^k = 0, \qquad h_{0m} = 1, \tag{11}$$

in which $m$ is a positive integer number and the $h_{jk}$ are constants.

In 2003 R. Conte and M. Musette have proposed a new method to find elliptic solutions [42]. This method is based on the Painlevé test and uses the Laurent series expansion to find the analytic form of elliptic solutions. Rather than substitute the first-order equation (11) into equation (8) one can substitute the found Laurent series of solutions of equation (8), for example, either solution (3) or solution (6), into equation (11) and obtain a linear system in $h_{jk}$. This method is more powerful than the traditional method and allows in principle to find all elliptic solutions. I hope that the use of this method allows finding of the three-parameter elliptic solutions.

## 5 Conclusion

Using the Painlevé analysis one can not only find integrable cases of dynamical systems, but also construct special solutions in nonintegrable cases.

We have found the special solutions of the Hénon–Heiles system with $C = -16/5$ and $C = -4/3$ as Laurent series depending on three parameters. For some values of these parameters the obtained solutions coincide with the known exact periodic solutions. There are no obstacles to existence of three-parameter single-valued solutions, so the probability of finding of exact three-parameter solutions that generalize the solutions obtained in [37, 39] is high.

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[1] Kowalevski S., Sur le problème de la rotation d'un corps solide autour d'un point fixe, *Acta Mathematica*, 1889, V.12, 177–232; Sur une properiété du sustème d'équations différentielles qui définit la rotation d'un corps solide autour d'un point fixe, *Acta Mathematica*, 1890, V.14, 81–93 (in French); reprinted in Kovalevskaya S.V., Scientific works, Moscow, AS USSR Publ. House, 1948 (in Russian).

[2] Golubev V.V., Lectures on the integration of the equation of motion of a heavy rigid body about a fixed point, Moscow, Gostekhizdat, 1953; reprinted Moscow–Izhevsk, RCD, 2002 (in Russian).

[3] Goriely A., A brief history of Kovalevskaya exponents and modern developments, *Regular and Chaotic Dynamics*, 2000, V.5, N 1, 3–16.

[4] Painlevé P., Leçons sur la théorie analytique des équations différentielles, profeesées à Stockholm (septembre, octobre, novembre 1895) sur l'invitation de S. M. le roi de Suède et de Norwège, Paris, Hermann, 1897; Reprinted in Œuvres de Paul Painlevé, V.1, Paris, ed. du CNRS, 1973; On-line version The Cornell Library Historical Mathematics Monographs, http://historical.library.cornell.edu/

[5] Golubev V.V., Lectures on analytical theory of differential equations, Moscow – Leningrad, Gostekhizdat, 1950 (in Russian).

[6] Hille E., Ordinary differential equations in the complex domain, New York, Wiley, 1976.

[7] Tabor M., Chaos and integrability in nonlinear dynamics, New York, Wiles, 1989 (in English); Moscow, URSS, 2001 (in Russian).

[8] Ercolani N. and Siggia E.D., Painlevé property and integrability, *Phys. Lett. A*, 1986, V.119, 112–116.

[9] Ercolani N. and Siggia E.D., Painlevé property and geometry, *Physica D*, 1989, V.34, 303–346.

[10] Kozlov V.V., Symmetry, topology and resonances in Hamiltonian mechanics, Izhevsk, UGU Publ. House, 1995 (in Russian); Berlin, Springer, 1995 (in English).

[11] Goriely A. and Tabor M., The role of complex-time singularities in chaotic dynamics, *Regular Chaotic Dynamics*, 1998, V.3, N 3, 32–44.

[12] Goriely A., Painlevé analysis and normal forms theory, *Physica D*, 2001, V.152–153, 124–144.

[13] Conte R. (Editor), The Painlevé property, one century later, Proceedings of the Cargèse school (3–22 June, 1996, Cargèse), CRM series in mathematical physics, Berlin, Springer–Verlag, 1998, New York, Springer, 1999.

[14] Ablowitz M.J., Ramani A. and Segur H., A connection between nonlinear evolution equations and ordinary differential equations of P-type. I, II, *J. Math. Phys.*, 1980, V.21, 715–721, 1006–1015.

[15] Weiss J., Tabor M. and Carnevale G., The Painlevé property for partial differential equations, *J. Math. Phys.*, 1983, V.24, 522–526.

[16] Weiss J., The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pairs and the Schwarzian derivative, *J. Math. Phys.*, 1983, V.24, 1405–1413.

[17] Conte R., Fordy A.P. and Pickering A., A perturbative Painlevé approach to nonlinear differential equations, *Physica D*, 1993, V.69, 33–58.

[18] Conte R., Universal invariance properties of Painlevé analysis and Bäcklund transformation in nonlinear partial differential equations, *Phys. Lett. A*, 1988, V.134, 100–104.

[19] Cariello F. and Tabor M., Painlevé expansions for nonintegrable evolution equations, *Physica D*, 1989, V.39, 77–94.

[20] Conte R., Exact solutions of nonlinear partial differential equations by singularity analysis, nlin.SI/0009024.

[21] Weiss J., Bäcklund transformation and linearizations of the Hénon–Heiles system, *Phys. Lett. A*, 1984, V.102, 329–331.

[22] Weiss J., Bäcklund transformation and the Hénon–Heiles system, *Phys. Lett. A*, 1984, V.105, 387–389.

[23] Gibbon J.D., Newell A.C., Tabor M. and Zeng Y.B., Lax pairs, Bäcklund transformation and special solutions for ordinary differential equations, *Nonlinearity*, 1988, V.1, 481–490.

[24] Sahadevan R., Painlevé expansion and exact solution for nonintegrable evolution equations, *Teor. Mat. Fiz.*, 1994, V.99, 528–536 (in Russian); translated in *Theor. Math. Phys.*, 1994, V.99, 796–802.

[25] Springael J., Conte R. and Musette M., On the exact solutions of the Bianchi IX cosmological model in the proper time, *Regular Chaotic Dynamics*, 1998, V.3, N 1, 3–8; nlin.SI/9804008.

[26] Hénon M. and Heiles C., The applicability of the third integral of motion: some numerical experiments, *Astronomical J.*, 1964, V.69, 73–79.

[27] Vernov S.Yu., The Painlevé analysis and special solutions for nonintegrable systems, math-ph/0203003.

[28] Podolský Ji. and Veselý K., Chaos in *pp*-wave spacetime, *Phys. Rev. D*, 1998, V.58, 081501.

[29] Kokubun F., Gravitational waves from the Hénon–Heiles system, *Phys. Rev. D*, 1998, V.57, 2610–2612.

[30] Antonowicz M. and Rauch-Wojciechowski S., Bi-Hamiltonian formulation of the Hénon–Heiles system and its multidimensional extensions, *Phys. Lett. A*, 1992, V.163, 167–172.

[31] Conte R., Musette M. and Verhoeven C., Integration of a generalized Hénon–Heiles Hamiltonian, *J. Math. Phys.*, 2002, V.43, 1906–1915; nlin.SI/0112030.

[32] Conte R., Musette M. and Verhoeven C., General solution for Hamiltonians with extended cubic and quadratic potentials, *Teor. Mat. Fiz.*, 2003, V.134, 148–159 (in Russian); translated in *Theor. Math. Phys.*, 2003, V.134, 128–138; nlin.SI/0301011.

[33] Melkonian S., Psi-series solutions of the cubic Hénon–Heiles system and their convergence, *J. Nonlinear Math. Phys.*, 1999, V.6, 139–160; math.DS/9904186.

[34] Hearn A.C., REDUCE. User's manual. Vers. 3.6, Santa Monica, California, RAND Publ. CP78, 1995; http://www.unikoeln.de/REDUCE/3.6/doc/reduce/
REDUCE. User's and Contributed Packages Manual, Vers. 3.7, Santa Monica, California, CA and Codemist Ltd, 1999; http://www.zib.de/Symbolik/reduce/more/moredocs/reduce.pdf

[35] Vernov S.Yu., Construction of solutions for the generalized Hénon–Heiles system with the help of the Painlevé test, *Teor. Mat. Fiz.*, 2003, V.135, 409–419 (in Russian); translated in *Theor. Math. Phys.*, 2003, V.135, 792–801; math-ph/0209063.

[36] Conte R. and Musette M., Link between solitary waves and projective Riccati equations, *J. Phys. A*, 1992, V.25, 5609–5623.

[37] Timoshkova E.I., A new class of trajectories of motion in the Hénon–Heiles potential field, *Rus. Astron. J.*, 1999, V.76, 470–475 (in Russian); translated in *Astron. Rep.*, 1999, V.43, 406–411.

[38] Timoshkova E.I., A new class of periodic orbits in the Henon–Heiles potential field, in Proceedings of the International Conference "Stellar Dynamics: from Classic to Modern" (21–27 August, 2000, Sankt-Petersburg, Russia), Editors L.P. Ossipkov and I.I. Nikiforov, Sankt-Petersburg, 2001, 201–205.

[39] Vernov S.Yu. and Timoshkova E.I., On two nonintegrable cases of the generalized Hénon–Heiles system with an additional nonpolynomial term, math-ph/0402049.

[40] Erdelyi A. et al. (Editors), Higher transcendental functions (based, in part, on notes left by H. Bateman), V.3, New York, London, MC Graw-Hill Book Comp., 1955 (in English); Moscow, Nauka, 1967 (in Russian).

[41] Briot C. and Bouquet T., Théorie des fonctions elliptiques. L. I-VIII, Paris, Gauthier-Villars, 1875.

[42] Conte R. and Musette M., Analytic solitary waves of nonintegrable equations, *Physica D*, 2003, V.181, 70–79; nlin.PS/0302051.