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Analytic Transformations between Surfaces with Animations

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We give a short description of our own software for geometry and differential geometry and its extensions [4, 1–3, 5–8] and apply it to the visualisation and animation of certain analytic transformations between surfaces.

1 Introduction

The main purpose of our software is to visualise the classical results in differential geometry on PC screens, plotters, printers or any other postscript device, but it also has extensions to physics, chemistry, crystallography and the engineering sciences. To the best of our knowledge, no other comparable, comprehensive software of this kind is available.

The software is *open* which means that its source files are accessible to the users, thus enabling them to apply it in the solutions of their own problems. This makes it extendable and flexible, and applicable to both teaching and research in many fields. In contrast to this, almost all other available graphics packages are *closed*; in general, the area below the user interface is inaccessible and consequently the software cannot be extended beyond the scope of solutions it offers. The software uses *OOP*, *object oriented programming*, and its programming language is *PASCAL*. The software is self-contained in the sense that no graphics package is needed other than PASCAL.

The advantages of PASCAL are the *hierarchy of objects* and the *polymorphy* which is not available in some *OOP languages*. In the hierarchy of objects, a successor inherits all the data, in particular the methods and procedures, of its predecessors. Polymorphy means that virtual methods can be declared, a virtual method can be rewritten with the same name in a successor, and one may have more methods than one with the same name. The development of our software could not have been achieved without OOP.

2 Spherical and pseudo-spherical surfaces of revolution

Let γ be a curve with a parametric representation $\vec{x}(s) = (r(s), 0, h(s))$ where $s \in I \subset \mathbb{R}$ is the arc length along γ, and $r(s) > 0$ on the interval I. Furthermore let $RS(\gamma)$ be the surface of revolution generated by rotating γ about the x^3-axis. Putting $u^1 = s$ and writing u^2 for the angle of rotation, we obtain the following parametric representation for $RS(\gamma)$ on $D = I \times (0, 2\pi)$

$$\vec{x}\big(u^i\big) = \big(r\big(u^1\big)\cos u^2, r\big(u^1\big)\sin u^2, h\big(u^1\big)\big). \tag{1}$$

Omitting the argument u^1, we find that the first and second fundamental coefficients of $RS(\gamma)$ are given by $g_{11} = (r')^2 + (h')^2 = 1$, since u^1 is the arc length along γ, $g_{12} = 0$, $g_{22} = r^2$, $L_{11} = r'h'' - r''h'$, $L_{12} = 0$ and $L_{22} = rh'$. So the Gaussian curvature

$$K = \frac{L}{g} = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2} \quad \text{of } RS(\gamma) \text{ is given by} \quad K = \frac{r'h'' - r''h'}{r}.$$

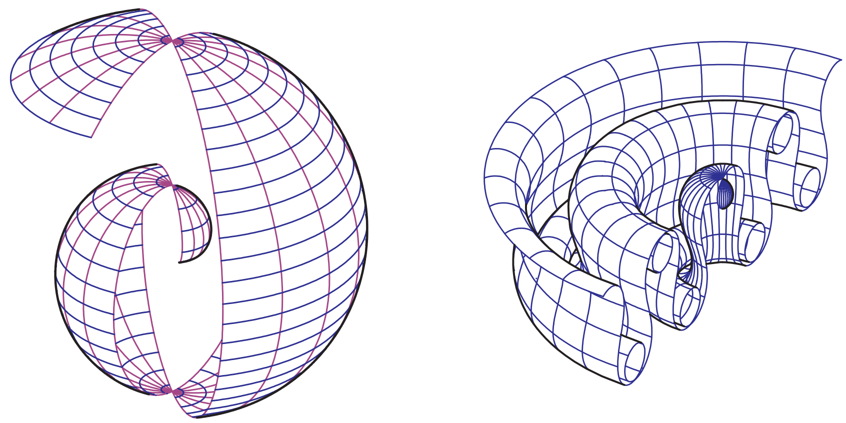

Figure 1. Surfaces of revolution.

Since $(r')^2 + (h')^2 = 1$ implies $r'r'' + h'h'' = 0$, we obtain

$$K = \frac{r'h''h' - r''(h')^2}{r} = -\frac{\big((r')^2 + (h')^2\big)r''}{r} = -\frac{r''}{r},$$

and consequently writing $u = u^1$

$$r''(u) + K(u)r(u) = 0. \tag{2}$$

Surfaces of revolution with constant Gaussian curvature $K > 0$ or $K < 0$ are called *spherical* or *pseudo-spherical surfaces of revolution*, respectively[1].

First, we assume $K = 0$. Then $r = c_1 u + c_2$ with constants c_1 and c_2. If we choose $c_1 = 0$ then $h' = \pm 1$ implies $h = \pm u + d$ with some constant d, and we obtain a circular cylinder. If $c_1 \neq 0$ then $(r')^2 + (h')^2 = 1$ implies $|c_1| \leq 1$. For $|c_1| = 1$, we have $h' \equiv 0$, hence $h \equiv \text{const}$, and we obtain a plane. For $0 < |c_1| < 1$ and a suitable choice of the coordinate system, we have $r = c_1 u$ and $h = d_1 u$ for some constant d_1 with $c_1^2 + d_1^2 = 1$, and we obtain a circular cone.

2.1 Spherical surfaces of revolution

We assume $K > 0$ and put $K = 1/c^2$ for some constant $c > 0$. Then the general solution of the differential equation (2) is $r(u) = \lambda \cdot \cos(u/c + u_0)$ and by a suitable choice of the arc length we may assume $r(u) = \lambda \cos(u/c)$ with $\lambda > 0$. Now $(r')^2 + (h')^2 = 1$ implies

$$h(u) = \pm \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\frac{u}{c}}\, du$$

and we may choose the upper sign without loss of generality. Thus the spherical surfaces of revolution are given by parametric representations (1) with

$$r\big(u^1\big) = \lambda \cos\frac{u^1}{c} \text{ and } h\big(u^1\big) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\frac{u^1}{c}}\, du^1,$$

where $\lambda > 0$, $c = \frac{1}{\sqrt{K}}$ and $K > 0$, the integral for h is a so-called elliptic integral. We obtain three different types of spherical surfaces of revolution corresponding to the cases $\lambda = c$, $\lambda > c$ or $\lambda < c$.

Case 1 $\lambda = c$. Then the surface has a parametric representation

$$\vec{x}\big(u^i\big) = \left(c\cos\frac{u^1}{c}\cos u^2, c\cos\frac{u^1}{c}\sin u^2, c\sin u^1\right).$$

Hence the surface is a sphere with radius c and centre in the origin.

[1]Figures in colour will be available only in electronic version.

Case 2 $\lambda > c$. The corresponding surfaces are called *hyperbolic spherical surfaces of revolution*. Now the integral for h only exists for values of u^1 that satisfy

$$\sin^2 \frac{u^1}{c} \leq \frac{c^2}{\lambda^2}, \text{ that is } u^1 \in I_k = \left[-c \arcsin \frac{c}{\lambda} + k\pi, c \arcsin \frac{c}{\lambda} + k\pi\right] \text{ for } k = 0, \pm 1, \pm 2, \ldots.$$

Every interval I_k defines a region of the surface. The radii of the circles of the u^2-lines are minimal at the end points of the intervals I_k and equal to $r = \sqrt{\lambda^2 - c^2}$, whereas the maximum radius $R = \lambda$ is attained in the middle of each region.

Case 3 $\lambda < c$. The corresponding surfaces are called *elliptic spherical surfaces of revolution*. Now the integral for h exists for all u^1 and the radii r of the circles of the u^2-lines attain all values $r \leq \lambda$.

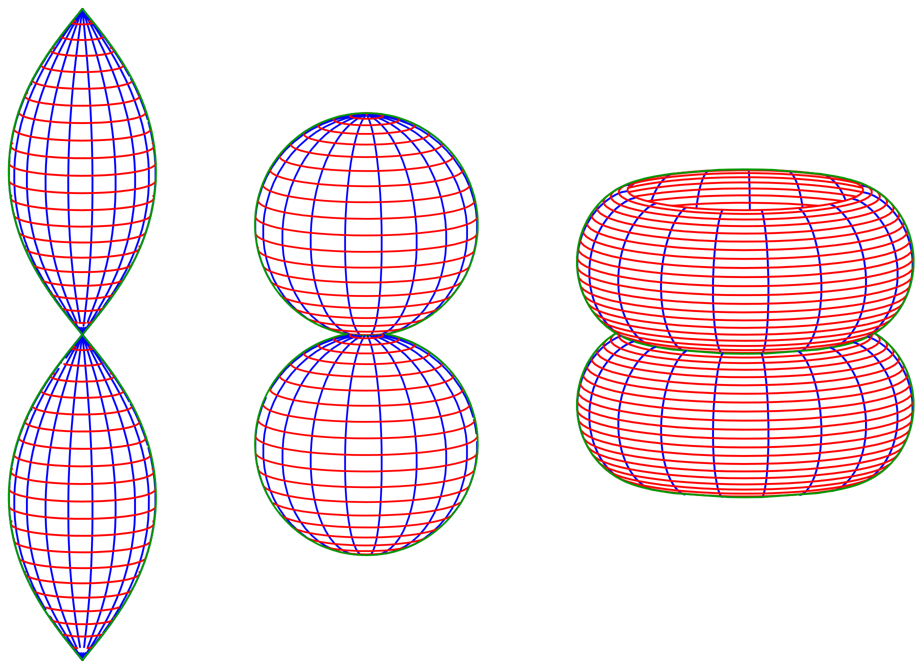

Figure 2. Spherical surfaces of revolution: $c = 1$, $\lambda = 0.6$, elliptic; $c = \lambda = 1$, spheres; $c = 1$, $\lambda = 1.4$, hyperbolic.

2.2 Pseudo-spherical surfaces of revolution

Now we assume $K < 0$ and put $K = -1/c^2$ for some constant $c > 0$. The general solution of the differential equation (2) is

$$r(u) = C_1 \cosh\left(\frac{u}{c}\right) + C_2 \sinh\left(\frac{u}{c}\right) \quad \text{with constants } C_1 \text{ and } C_2. \tag{3}$$

Case 1 $C_1 = -C_2 = \lambda \neq 0$. Then we obtain from (3) $r(u) = \lambda \exp(-u/c)$, and the surface has a parametric representation with

$$r(u^1) = \lambda \exp\left(-\frac{u^1}{c}\right) \quad \text{and} \quad h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \exp\left(-\frac{2u^1}{c}\right)}\, du.$$

Such a surface is called *parabolic pseudo-spherical surface of revolution*. The integral for h exists for $|u^1| > c \log(|\lambda|/c)$.

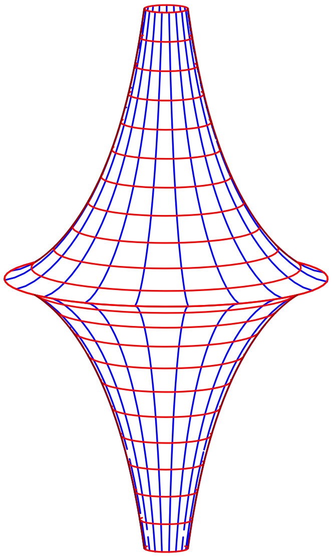

Figure 3. Parabolic pseudo-spherical surfaces of revolution ($\lambda = 1$): $c = 1$; $c = 1.5$; $c = 2$.

Case 2 $C_2 = 0$ and $C_1 = \lambda \neq 0$. Then the surface has a parametric representation with

$$r(u^1) = \lambda \cosh\left(\frac{u^1}{c}\right) \quad \text{and} \quad h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\sinh^2\left(\frac{u^1}{c^2}\right)}\, du^1.$$

Such a surface is called *hyperbolic pseudo-spherical surface of revolution*. The integral for h is an elliptic integral and exists for $|u^1| \leq c \cdot \operatorname{Arsh}(c/|\lambda|) = c \log\left(c/|\lambda| + \sqrt{1 + c^2/\lambda^2}\right)$. The radii r of the circles of the u^2-lines satisfy $\lambda \leq r \leq \sqrt{\lambda^2 + c^2}$.

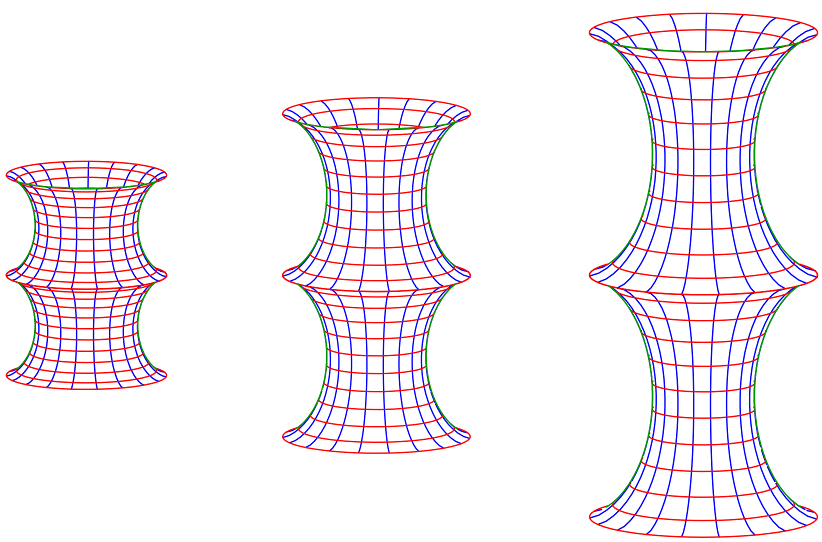

Figure 4. Hyperbolic pseudo-spherical surfaces of revolution ($\lambda = 1$): $c = 1.2$; $c = 1.6$; $c = 2$.

Case 3 $C_1 = 0$ and $C_2 = \lambda \neq 0$. Then the surface has a parametric representation with

$$r(u^1) = \lambda \sinh\left(\frac{u^1}{c}\right) \quad \text{and} \quad h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\cosh^2\left(\frac{u^1}{c^2}\right)}\, du^1.$$

Such a surface is called *elliptic pseudo-spherical surface of revolution*. Since $\cosh x \geq 1$ for all x, we must have $|\lambda| \leq c$. The integral for h is elliptic and exists for $\cosh\left(u^1/c\right) \leq c/|\lambda|$. The radii r of the circles of the u^2-lines satisfy $0 \leq r \leq \sqrt{c^2 - \lambda^2}$.

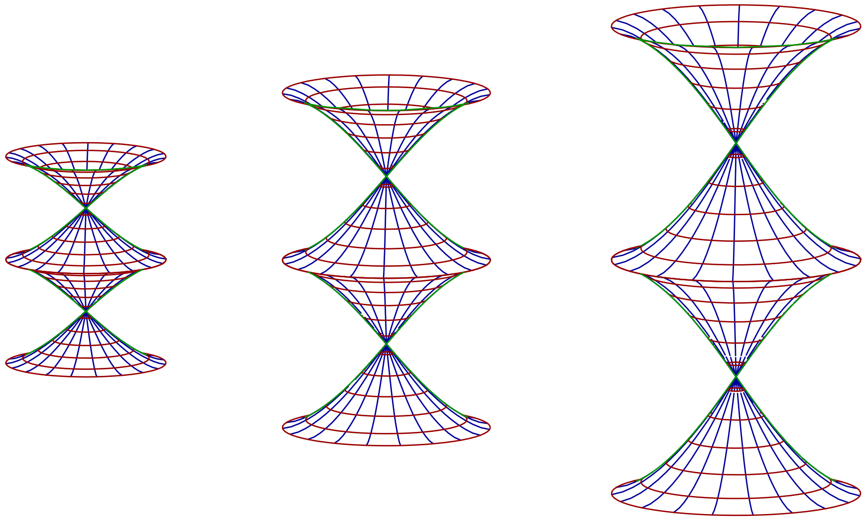

Figure 5. Elliptic pseudo-spherical surfaces of revolution ($\lambda = 1$): $c = 1.35$; $c = 1.55$; $c = 1.75$.

3 Isometric mappings of surfaces of revolution

Now we determine all surfaces of revolution that can be mapped isometrically onto a surface of revolution with non-constant Gaussian curvature.

Since the first and second fundamental coefficients of surfaces of revolution S depend on the parameter u^1 only, the Gaussian curvature is constant along any u^2-line of S. Furthermore the Gaussian curvature of surfaces is invariant under isometric mappings. Since the Gaussian curvature of the given surface of revolution S is not constant, the u^2-lines of the surfaces of revolution S^* that are to be determined have to be mapped onto the u^2-lines of S. Finally, since isometric maps are conformal, the u^1-lines of S^* have to be mapped onto the u^1-lines of S. Let S and S^* be given by parametric representations (1) and

$$\vec{x}(u^{*i}) = \left(r^*(u^{*1})\cos u^{*2}, r^*(u^{*1})\sin u^{*2}, h^*(u^{*1})\right),$$

where u^1 and u^{*1} are the arc lengths along the u^1-lines of S and the u^{*1}-lines of S^*, respectively, that is $\left(r'(u^1)\right)^2 + \left(h'(u^1)\right)^2 = 1$ and $\left(r^{*'}(u^{*1})\right)^2 + \left(h^{*'}(u^{*1})\right)^2 = 1$. The first fundamental coefficients are $g_{11} = g^*_{11} = 1$, $g_{12} = g^*_{12} = 0$ and $g_{22} = r(u^1)$ and $g^*_{22} = r^*(u^{*1})$, hence the first fundamental forms are

$$(ds)^2 = \left(du^1\right)^2 + r^2(u^1)\left(du^2\right)^2 \quad \text{and} \quad (ds^*)^2 = \left(du^{*1}\right)^2 + \left(r^*(u^{*1})\right)^2\left(du^{*2}\right)^2. \tag{4}$$

Since the u^1- and u^2-lines of S correspond to the u^{*1}- and u^{*2}-lines of S^*, we must have

$$u^{*1}(u^1, u^2) = u^{*1}(u^1) \quad \text{and} \quad u^{*2}(u^1, u^2) = u^{*2}(u^2).$$

The first fundamental coefficients of isometric surfaces must satisfy $g_{ik} = g^*_{ik}$ for $i, k = 1, 2$. Therefore we obtain from (4)

$$du^{*1} = \pm du^1 \quad \text{and} \quad \frac{r(u^1)}{r^*(u^1)} = \pm\frac{du^{*2}}{du^2}. \tag{5}$$

First, the left-hand side of (5) implies $u^{*1} = \pm u^1 + c_0$ where c_0 is a constant. If we choose the same orientation of the u^1 and u^{*1}-lines, we may assume $u^{*1} = u^1$. Then the right hand side of (5) becomes

$$\frac{r(u^1)}{r^*(u^1)} = \pm\frac{du^{*2}(u^2)}{du^2}.$$

Since the left-hand side of this identity only depends on u^1 and the right hand side only on u^2, we must have

$$r^*(u^1) = c \cdot r(u^1) \quad \text{and} \quad \frac{du^{*2}}{du^2} = \pm\frac{1}{c} \quad \text{where } c \neq 0 \text{ is a constant.}$$

This implies $u^{*2} = \pm u^2/c + d$, where d is a constant. We may choose $d = 0$ and the upper sign, since d corresponds to a certain rotation about the axis and the sign to a reflection. Thus we obtain $u^{*1} = u^1$ and $u^{*2} = u^2/c$. Since u^{*1} is the arc length along the u^*-lines, it follows that

$$h^{*'}(u^{*1}) = \sqrt{1 - \left(r^{*'}(u^{*1})\right)^2} = \sqrt{1 - c^2\left(r'(u^1)\right)^2}.$$

Consequently every surface of revolution S^* that can be mapped isometrically onto a given surface of revolution S with non–constant Gaussian curvature is given by a parametric representation

$$\vec{x}^*(u^i) = \left(cr(u^1)\cos\frac{u^2}{c}, cr(u^1)\sin\frac{u^2}{c}, h^*(u^1)\right) \quad \text{with} \quad h^*(u^1) = \int\sqrt{1 - c^2\left(r'(u^1)\right)^2}du^1.$$

If $c = 1$ then $S^* = S$. A continuous change of c corresponds to a continuous deformation. For a given surface of rotation there is a one parametric family of isometric surfaces of revolution.

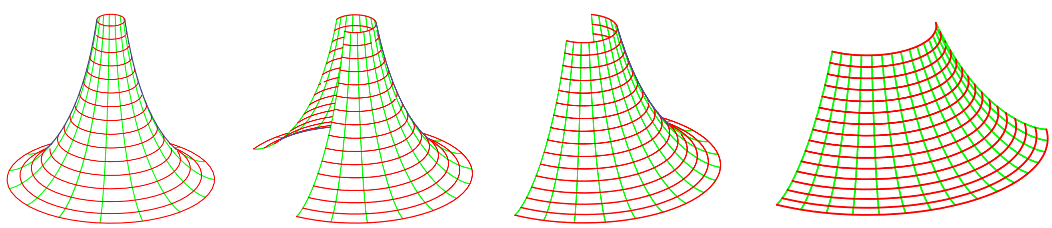

Figure 6. An isometric deformation of a pseudo-sphere.

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