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On the Moments of the Traces of Unitary and Orthogonal Random Matrices

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The complicated moments of the traces of unitary and orthogonal Haar distributed random matrices are studied. The exact formulas for different values of moment and matrix orders are obtained.

1 Introduction

Consider the probability space, whose objects are $n \times n$ unitary matrices, and whose probability measure is unit Haar measure on the group $U(n)$. Denote $\langle \cdot \rangle$ the expectation with respect to the measure. Given a non-negative integer k, consider the moment

$$\left\langle (\mathrm{Tr}\, U_n)^{a_1} (\mathrm{Tr}\, U_n^2)^{a_2} \cdots (\mathrm{Tr}\, U_n^k)^{a_k} \overline{(\mathrm{Tr}\, U_n)^{b_1}} \cdots \overline{(\mathrm{Tr}\, (U_n^k)^{b_k}} \right\rangle, \tag{1}$$

where $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ are k-tuples of non-negative integers. If

$$\kappa(a) = \sum_{j=1}^{k} j a_j, \qquad \kappa(b) := \sum_{j=1}^{k} j b_j, \tag{2}$$

and $\kappa(a) \neq \kappa(b)$, then it is easy to see that (1) is equal to zero. Hence, without loss of generality, we can restrict ourselves to the moments for which $\kappa(a) = \kappa(b)$. In this case we call

$$\kappa := \kappa(a) = \kappa(b) \tag{3}$$

the order of the respective moment, and we write

$$m_\kappa^{(n)}(a; b) = \left\langle (\mathrm{Tr}\, U_n)^{a_1} (\mathrm{Tr}\, U_n^2)^{a_2} \cdots (\mathrm{Tr}\, U_n^k)^{a_k} \overline{(\mathrm{Tr}\, U_n)^{b_1}} \cdots \overline{(\mathrm{Tr}\, U_n^k)^{b_k}} \right\rangle, \tag{4}$$

In recent paper [3] Diaconis and Evans proved that if $\kappa \leq n$, then

$$m_\kappa^{(n)}(a; b) = \delta_{a,b} \prod_{j=1}^{k} j^{a_j} a_j!, \qquad \delta_{a,b} = \prod_{j=1}^{k} \delta_{a_j, b_j}, \tag{5}$$

Analogous result was obtained for the orthogonal group also (see formula (17) below). The proofs of (5) in [3] were based on the representation theory of the groups $U(n)$ and $O(n)$. Another proof of (5) was given in [6] (Appendix). The proof is based on certain identities for the Töplitz determinants [1] Hughes and Rudnick [5] proved relation (17) for $SO(n)$ and for $\kappa \leq n-1$ using some combinatorial techniques. Similar questions were considered in [2, 4, 9, 11].

We will give below a proof of these relations for the groups $U(n)$ and $O(n)$, by using elementary means.

2 Unitary group

We begin from (5). Our proof is based on the following simple implication of the left invariance of Haar measure on $U(n)$.

Proposition 1. *Let $F(U_n, U_n^*)$ be a continuously differentiable complex valued function. Then for any $n \times n$ Hermitian matrix H we have*

$$\left\langle F_1'(U_n, U_n^*) \cdot HU_n - F_2'(U_n, U_n^*) \cdot U_n^* H \right\rangle = 0, \tag{6}$$

where $F_1'(U_n, U_n^*)$ and $F_2'(U_n, U_n^*)$ are derivatives of F with respect to U_n and U_n^* correspondingly.

The proposition follows from the fact that $\left\langle F\left(e^{itH}U_n, U_n^* e^{-itH}\right)\right\rangle$ is independent of a real parameter t because of the left invariance of Haar measure on $U(n)$.

Choosing in (6) $H = zX^{(x,y)} + \overline{z}\left(X^{(x,y)}\right)^T$, where z is an arbitrary complex number and $X^{(x,y)} = \{\delta_{px}\delta_{qy}\}_{p,q=1}^n$, we conclude that formula (6) is valid also in the case where H is replaced by $X^{(x,y)}$ (in fact, by any real matrix).

Denote a_j the first from the left non-zero index of the k-tuple $a = (0, \ldots, 0, a_j, \ldots, a_k)$. Then we can write (4) as

$$m_k^{(n)}(a; b) = \sum_{x=1}^{n} \left\langle (U_n^j)_{x,x} (\operatorname{Tr} U_n^j)^{a_j - 1} \cdots (\operatorname{Tr} U_n^k)^{a_k} (\operatorname{Tr} U_n^*)^{b_1} \cdots (\operatorname{Tr} (U_n^*)^k)^{b_k} \right\rangle . \tag{7}$$

We apply the proposition with $H = X^{(x,y)}$ to the function

$$F(U_n, U_n^*) = (U_n^j)_{x,y} (\operatorname{Tr} U_n^j)^{a_j-1} \cdots (\operatorname{Tr} U_n^k)^{a_k} (\operatorname{Tr} U_n^*)^{b_1} \cdots (\operatorname{Tr} (U_n^*)^k)^{b_k}.$$

Taking into account the relations

$$\begin{aligned}
((U_n^m)_{x,y})' X^{(x,y)} U_n &= \sum_{i=0}^{m-1} (U_n^i X^{(x,y)} U_n^{m-i})_{x,y} = \sum_{i=0}^{m-1} (U_n^i)_{x,x} (U_n^{m-i})_{y,y} \\
&= \delta_{x,x} (U_n^m)_{y,y} + \sum_{i=1}^{m-1} (U_n^i)_{x,x} (U_n^{m-i})_{y,y}, \\
(\operatorname{Tr} U_n^m)' \cdot X^{(x,y)} U_n = m \operatorname{Tr} U_n^m X &= m (U_n^m)_{y,x}, \qquad (\operatorname{Tr} (U_n^*)^m)' X^{(x,y)} U_n^* = m((U_n^*)^m)_{y,x},
\end{aligned}$$

and similar ones, we obtain

$$\begin{aligned}
&\delta_{xx} \left\langle (U_n^j)_{y,y} \alpha_- \beta \right\rangle + \sum_{i=1}^{j-1} \left\langle (U_n^i)_{x,x} (U_n^{j-i})_{y,y} \alpha_- \beta \right\rangle \\
&\quad + (a_j - 1) j \left\langle (U_n^j)_{x,y} (U_n^j)_{y,x} (\operatorname{Tr} U_n^j)^{a_j - 2} \alpha_+ \beta \right\rangle \\
&\quad + \sum_{l=j+1}^{k} a_l l \left\langle (U_n^j)_{x,y} (U_n^l)_{y,x} \alpha(l) \beta \right\rangle - \sum_{l=1}^{k} b_l l \left\langle (U_n^j)_{x,y} ((U_n^*)^l)_{y,x} \alpha \beta(l) \right\rangle = 0,
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
&\alpha_- = (\operatorname{Tr} U_n^j)^{a_j-1} \cdots (\operatorname{Tr} U_n^k)^{a_k}, \qquad \beta = (\operatorname{Tr} U_n^*)^{b_1} \cdots (\operatorname{Tr} (U_n^*)^k)^{b_k}, \\
&\alpha_+ = (\operatorname{Tr} U_n^{j+1})^{a_{j+1}} \cdots (\operatorname{Tr} U_n^k)^{a_k}, \qquad \alpha(l) = (\operatorname{Tr} U_n^j)^{a_j-1} \cdots (\operatorname{Tr} U_n^l)^{a_l-1} \cdots (\operatorname{Tr} U_n^k)^{a_k}, \\
&\beta(l) = (\operatorname{Tr} U_n^*)^{b_1} \cdots (\operatorname{Tr} (U_n^*)^l)^{b_l-1} \cdots (\operatorname{Tr} (U_n^*)^k)^{b_k}.
\end{aligned}$$

Applying to (8) the operation $n^{-1}\sum\limits_{x,y=1}^{n}$ and regrouping terms we obtain in view of (7)

$$\begin{aligned}
&m_\kappa^{(n)}(a;b) + \frac{1}{n}\Bigg(\sum_{l=1}^{j-1} m_\kappa^{(n)}(0,\dots,a_l=1,0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k;b) \\
&\quad + j(a_j-1)m_\kappa^{(n)}(0,\dots,0,a_j-2,a'(2j),a_{2j}+1,a''(2j);b) \\
&\quad + \sum_{l=j+1}^{k} l a_l m_\kappa^{(n)}(0,\dots,0,a_j-1,a'(l),a_l-1,\tilde{a},a_{l+j}+1,a''(l+j);b)\Bigg) \\
&= j b_j m_{\kappa-j}^{(n)}(0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(j),b_j-1,b''(j)) \\
&\quad + \frac{1}{n}\Bigg(\sum_{l=1}^{j-1} l b_l m_{\kappa-l}^{(n)}(0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(l),b_l-1,b''(l)) \\
&\quad + \sum_{l=j+1}^{k} l b_l m_{\kappa-j}^{(n)}(0,\dots,a_j-1,a';b'(l-j),b_{l-j}+1,\tilde{b},b_l-1,b''(l))\Bigg), \qquad (9)
\end{aligned}$$

where

$$\begin{aligned}
a'(l) &= (a_{j+1},\dots,a_{l-1}), \quad & a''(l) &= (a_{l+1},\dots,a_k), \quad & \tilde{a} &= (a_{l+1},\dots,a_{l+j-1}), \\
b'(l) &= (b_1,\dots,b_{l-1}), \quad & b''(l) &= (b_{l+1},\dots,b_k), \quad & \tilde{b} &= (b_{l-j+1},\dots,b_{l-1}).
\end{aligned}$$

It will be important in what follows that the order of all moments in the l.h.s. of (9) is κ, while the orders of all moments in the l.h.s. are less then κ ($\kappa - j$ and $\kappa - l$).

We present now these relations in a more convenient form. Given a non-negative integer K, denote P_K the set of the k-tuples $a = (a_1,\dots,a_k)$, $a_j \ge 0$ such that

$$\sum_{j=1}^{k} j a_j \le K.$$

Consider the vector space $\mathcal{L}_K$ of collections of complex numbers, indexed by pairs $(a;b)$ of k-tuples such that

$$a,\, b \in P_K, \qquad \sum_{j=1}^{k} j a_j = \sum_{j=1}^{k} j b_j \le K, \qquad (10)$$

and call the integer κ of (3) the order of a component $v(a;b)$ of $v \in \mathcal{L}_K$, if the indices a and b of the component satisfy (3). We define in $\mathcal{L}_K$ the $\|\cdot\|_\infty$-norm, i.e. if $v \in \mathcal{L}_K$, then

$$\|v\| = \max_{a,b} |v(a;b)|,$$

where the maximum is taken over the pairs (a,b), satisfying (10). Furthermore, we view the expression in the parentheses of the l.h.s. of (9), the first term of the r.h.s., and the expression in the parentheses of the r.h.s. of (9) as the results of action of the linear operators A, B, and C on the vector $m_K^{(n)}$, whose components are the moments (4) of the orders $\kappa \le K$. In other words, if a_j is the first from the left non-zero component of the k-tuple a, then

$$(Av)_\kappa(a;b) = \frac{1}{n}\Bigg(\sum_{l=1}^{j-1} v_\kappa(0,\dots,a_l=1,0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k;b) \qquad (11)$$

$$+ j(a_j-1)v_\kappa(0,\dots,0,a_j-2,a'(2j),a_{2j}+1,a''(2j);b)$$
$$\left. + \sum_{l=j+1}^{k} la_l v_\kappa(0,\dots,0,a_j-1,a'(l),a_l-1,\tilde{a},a_{l+j}+1,a''(l+j);b)\right),$$

$$(Bv)_\kappa(a;b) = jb_j v_{\kappa-j}(0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(j),b_j-1,b''(j)), \tag{12}$$

$$(Cv)_\kappa(a;b) = \frac{1}{n}\left(\sum_{l=1}^{j-1} lb_l v_{\kappa-l}(0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(l),b_l-1,b''(l))\right.$$
$$\left. + \sum_{l=j+1}^{k} lb_l v_{\kappa-j}(0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(l-j),b_{l-j}+1,\tilde{b},b_l-1,b''(l))\right). \tag{13}$$

With this notation we can rewrite (9) as

$$(I+A)\, m_K^{(n)} = Bm_K^{(n)} + Cm_K^{(n)}. \tag{14}$$

On other hand, if we denote as $\mu_\kappa(a;b)$ the r.h.s. of (5) for $\kappa(a)=\kappa$, then we can easily obtain that the values $\mu_\kappa(a;b)$ verify the following recursive relation for any $a_j \neq 0$

$$\mu_\kappa(a;b) = jb_j\mu_{\kappa-j}(a_1,\dots,a_{j-1},a_j-1,a''(j);b'(j),b_j-1,b''(j)). \tag{15}$$

By using (11)–(15), it is easy to prove the following

Lemma 1. *Let A, B and C be the linear operators, defined by (11)–(13). Then*
(i) $||A|| \le (K-1)/n$,
(ii) if μ_K is the vector of $\mathcal{L}_K$, whose components are $\mu_\kappa(a;b)$, $\kappa \le K$, then

$$B\mu_K = \mu_K, \qquad C\mu_K = A\mu_K.$$

Remark 1. Relation (15) (and hence its solution (5)) can be also obtained form (14). Indeed, we obtain (15) just passing in (14) to the limit $n \to \infty$ formally. To justify this, using the first assertion of the lemma and induction in K we can prove that the sequence $\{m_K^{(n)}\}_{n=1}^\infty$ is uniformly in n bounded and hence compact.

The first assertion of the lemma implies that if $K \le n$, then the operator $I+A$ is invertible. Hence, for $K \le n$ (14) (or (9)) is equivalent to

$$m_K^{(n)} = (I+A)^{-1}\left(Bm_K^{(n)} + Cm_K^{(n)}\right).$$

Since $Bm_K^{(n)}$ and $Cm_K^{(n)}$ include the moments whose orders are strictly smaller than K, this relation allows us to find the moments of the order K, provided that the moments of lower orders are known. This suggests the use of the induction in K to prove formula (5).

Indeed, it is easy to check that for $K=0,1$ the formula (5) holds. Assume that $m_\kappa^{(n)}(a,b) = \mu_\kappa(a,b)$, $\forall\, \kappa \le K-1$. The r.h.s. of (12)–(13) contain the components of v, whose orders do not exceed $K-1$. Hence $Bm_K^{(n)} = B\mu_K$, $Cm_K^{(n)} = C\mu_K$, $(B+C)m_K^{(n)} = (I+A)\mu_K$. These facts and the second assertion of the lemma yield the following relation

$$m_K^{(n)} = (I+A)^{-1}\left(B\mu_K + C\mu_K\right) = (I+A)^{-1}(I+A)\mu_K = \mu_K,$$

which complete the proof of (5).

3 Orthogonal group

Consider now the probability space, whose objects are $n \times n$ orthogonal matrices, and whose probability measure is the normalized to unit Haar measure on the group $O(n)$ and denote $\langle \cdot \rangle$ the expectation with respect to the measure. Given a non-negative integer k, consider the moment

$$m_\kappa^{(n)}(a) = \left\langle (\operatorname{Tr} O_n)^{a_1} (\operatorname{Tr} O_n^2)^{a_2} \cdots (\operatorname{Tr} O_n^k)^{a_k} \right\rangle, \tag{16}$$

where $a = (a_1, \ldots, a_k)$ is k-tuple.

In recent paper [3] Diaconis and Evans proved that if $\kappa \le n/2$, then

$$m_\kappa^{(n)}(a) = \mathbb{E}\left[\prod_{j=1}^{k} \left(\sqrt{j}\xi_j + y_j\right)^{a_j}\right], \qquad y_j = \left(1 + (-1)^j\right)/2, \tag{17}$$

and ξ_j are i.i.d. standard normal variables.

Hughes and Rudnick [5] we proved (17) for $SO(n)$ and $\kappa \le n-1$ using some combinatorial technic.

As in the previous section we will use the following proposition

Proposition 2. *Let $F(O_n)$ be a continuously differentiable complex-valued function. Then for any $n \times n$ real antisymmetric matrix X we have (cf.* (6)*)*

$$\left\langle F'(O_n) \cdot X O_n \right\rangle = 0, \tag{18}$$

where $F'(O_n)$ is the derivative of F with respect to O_n.

The proposition follows from the fact that $\left\langle F(e^{tX} O_n) \right\rangle$ is independent of a real parameter t.

Denote a_j the first from the left non-zero index of the k-tuple $a = (0, \ldots, 0, a_j, \ldots, a_k)$. Then we can write (16) as

$$m_k^{(n)}(a) = \sum_{x=1}^{n} \left\langle (O_n^j)_{x,x} (\operatorname{Tr} O_n^j)^{a_j - 1} \cdots (\operatorname{Tr} O_n^k)^{a_k} \right\rangle. \tag{19}$$

We apply the proposition with $X^{(x,y)} = \{\delta_{px}\delta_{qy} - \delta_{py}\delta_{qx}\}_{p,q=1}^n$ to the function

$$F(O_n) = (O_n^j)_{x,y} (\operatorname{Tr} O_n^j)^{a_j - 1} \cdots (\operatorname{Tr} O_n^k)^{a_k}.$$

Taking into account the relations

$$\begin{aligned}
((O_n^m)_{x,y})' X^{(x,y)} O_n &= \sum_{i=0}^{m-1} \left(O_n^i X^{(x,y)} O_n^{m-i}\right)_{x,y} \\
&= \sum_{i=0}^{m-1} \left((O_n^i)_{x,x} (O_n^{m-i})_{y,y} - (O_n^i)_{x,y} (O_n^{m-i})_{x,y}\right) \\
&= \delta_{x,x} (O_n^m)_{y,y} + \sum_{i=1}^{m-1} (O_n^i)_{x,x} (O_n^{m-i})_{y,y} - \delta_{x,y} (O_n^m)_{x,y} - \sum_{i=1}^{m-1} (O_n^i)_{x,y} \left((O_n^{m-i})^T\right)_{y,x}, \\
(\operatorname{Tr} O_n^m)' \cdot X^{(x,y)} O_n &= m \operatorname{Tr} O_n^m X^{(x,y)} = m (O_n^m)_{y,x} - m (O_n^m)_{x,y} \\
&= m (O_n^m)_{y,x} - m \left((O_n^m)^T\right)_{y,x},
\end{aligned}$$

and similar ones, we obtain

$$\begin{aligned}
&\delta_{xx}\left\langle (O_n^j)_{y,y}\alpha\right\rangle - \delta_{xy}\left\langle (O_n^j)_{x,y}\alpha\right\rangle + \sum_{i=1}^{j-1}\left\langle (O_n^i)_{x,x}(O_n^{j-i})_{y,y}\alpha_-\right\rangle \\
&\quad - \sum_{i=1}^{j-1}\left\langle (O_n^i)_{x,y}(O_n^{j-i})^T_{y,x}\alpha_-\right\rangle + (a_j-1)j\left(\left\langle (O_n^j)_{x,y}(O_n^j)_{y,x}(\operatorname{Tr} O_n^j)^{a_j-2}\alpha_+\right\rangle\right. \\
&\quad \left. - \left\langle (O_n^j)_{x,y}(O_n^j)^T_{y,x}(\operatorname{Tr} O_n^j)^{a_j-2}\alpha_+\right\rangle\right) \\
&\quad + \sum_{l=j+1}^{k} a_l l\left\langle (O_n^j)_{x,y}(O_n^l)_{y,x}\alpha(l)\right\rangle - a_l l\left\langle (O_n^j)_{x,y}(O_n^l)^T_{y,x}\alpha(l)\right\rangle = 0,
\end{aligned} \tag{20}$$

where

$$\alpha_- = (\operatorname{Tr} O_n^j)^{a_j-1}\cdots(\operatorname{Tr} O_n^k)^{a_k}, \qquad \alpha_+ = (\operatorname{Tr} O_n^{j+1})^{a_{j+1}}\cdots(\operatorname{Tr} O_n^k)^{a_k},$$
$$\alpha(l) = (\operatorname{Tr} O_n^j)^{a_j-1}\cdots(\operatorname{Tr} O_n^l)^{a_l-1}\cdots(\operatorname{Tr} O_n^k)^{a_k},$$

Applying to (20) the operation $n^{-1}\sum_{x,y=1}^{n}$ and regrouping terms we obtain in view of (19)

$$\begin{aligned}
&m_\kappa^{(n)}(a) + \frac{1}{n-1}\left(\sum_{l=1}^{j-1} m_\kappa^{(n)}(0,\ldots,a_l=1,0,\ldots,a_{j-l}=1,0,\ldots,a_j-1,a_{j+1},\ldots,a_k)\right. \\
&\quad + j(a_j-1)m_\kappa^{(n)}(0,\ldots,0,a_j-2,a'(2j),a_{2j}+1,a''(2j)) \\
&\quad \left. + \sum_{l=j+1}^{k} l a_l m_\kappa^{(n)}(0,\ldots,0,a_j-1,a'(l),a_l-1,\tilde a,a_{l+j}+1,a''(l+j))\right) \\
&= \frac{n}{n-1}\left(y_j m_{\kappa-j}^{(n)}(0,\ldots,a_j-1,a_{j+1},\ldots,a_k) + j(a_j-1)m_{\kappa-2j}^{(n)}(0,\ldots,a_j-2,a_{j+1},\ldots,a_k)\right) \\
&\quad + \frac{1}{n-1}\left(2\sum_{i<j/2} m_{\kappa-2i}^{(n)}(0,\ldots,a_{j-2i}=1,0,\ldots,a_j-1,a_{j+1},\ldots,a_k)\right. \\
&\quad \left. + \sum_{l=j+1}^{k} l a_l m_{\kappa-2j}^{(n)}(0,\ldots,a_j-1,a'(l-j),a_{l-j}+1,\hat a,a_l-1,a''(l))\right),
\end{aligned} \tag{21}$$

where y_j is defined in (17)

$$a'(l) = (a_{j+1},\ldots,a_{l-1}), \qquad a''(l) = (a_{l+1},\ldots,a_k),$$
$$\tilde a = (a_{l+1},\ldots,a_{l+j-1}), \qquad \hat a = (a_{l-j+1},\ldots,a_{l-1}).$$

As in the previous section we consider the vector space $\mathcal{L}_K$ of collections of complex numbers, indexed by k-tuples a such that $\sum_{j=1}^{k} j a_j \le K.$ and call the integer κ of (3) the order of a component $v(a)$ of $v \in \mathcal{L}_K$, if the index a of the component satisfy (3). We define in $\mathcal{L}_K$ the $\|\cdot\|_\infty$-norm, i.e. if $v \in \mathcal{L}_K$, then

$$\|v\| = \max_{a\in P_K} |v(a)|.$$

Furthermore, we view the expression in the parentheses of the l.h.s. of (21), the first term of the r.h.s., and the expression in the parentheses of the r.h.s. of (21) as the results of action of the linear operators A, B, and C on the vector $m_K^{(n)}$, whose components are the moments (16) of the orders $\kappa \le K$. In other words, if a_j is the first from the left non-zero component of the k-tuple a, then

$$(Av)_\kappa(a) = \frac{1}{n-1}\left(\sum_{l=1}^{j-1} \nu_\kappa(0,\dots,a_l=1,0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k)\right.$$
$$+ j(a_j-1)v_\kappa(0,\dots,0,a_j-2,a'(2j),a_{2j}+1,a''(2j))$$
$$\left.+ \sum_{l=j+1}^{k} la_l v_\kappa(0,\dots,0,a_j-1,a'(l),a_l-1,\tilde a,a_{l+j}+1,a''(l+j))\right), \tag{22}$$

$$(Bv)_\kappa(a) = y_j\nu_{\kappa-j}(0,\dots,a_j-1,a_{j+1},\dots,a_k)$$
$$+ j(a_j-1)\nu_{\kappa-2j}(0,\dots,a_j-2,a_{j+1},\dots,a_k), \tag{23}$$

$$(Cv)_\kappa(a) = \frac{1}{n-1}\left(2\sum_{i<j/2} \nu_{\kappa-2i}(0,\dots,a_{j-2i}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k)\right.$$
$$+ y_j\nu_{\kappa-j}(0,\dots,a_j-1,a_{j+1},\dots,a_k)$$
$$+ j(a_j-1)\nu_{\kappa-2j}(0,\dots,a_j-2,a_{j+1},\dots,a_k)$$
$$\left.+ \sum_{l=j+1}^{k} la_l\nu_{\kappa-2j}(0,\dots,a_j-1,a'(l-j),a_{l-j}+1,\hat a,a_l-1,a''(l))\right). \tag{24}$$

With this notation we can rewrite (21) as

$$(I+A)\,m_K^{(n)} = Bm_K^{(n)} + Cm_K^{(n)}. \tag{25}$$

On other hand, if we denote as $\mu_\kappa(a)$ the r.h.s. of (17) then, using Novikov–Furutzu formula for the average of ξ_j, we obtain for any $a_j \ne 0$ the following recursive relation

$$\mu_\kappa(a) = y_j\mu_{\kappa-j}(a_1,\dots,a_{j-1},a_j-1,a''(j))$$
$$+ j(a_j-1)\mu_{\kappa-2j}(a_1,\dots,a_{j-1},a_j-2,a''(j)). \tag{26}$$

As in the previous section, using (22)–(26), it is easy to prove the Lemma 1 with the only modification of its first assertion $||A|| \le (K-1)/(n-1)$. Thus, the rest of the proof coincides with unitary case.

It must be noted that we can also obtain relation (26) by passing to the limit $n \to \infty$ in (25).

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