Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 1, 498–503

Higher Conserved Quantities for the Multi-Centre Metrics

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The multi-centre metrics are a family of Euclidean solutions of the empty space Einstein equations with self-dual curvature. We consider the Hamiltonian dynamics generated by the geodesic flow along the $U(1)$ fibers and obtain several metrics in this class which do exhibit an extra conserved quantity quadratic in the momenta. The corresponding systems are shown to be classically integrable.

1 Introduction

The integrability of the geodesic flow for the multi-centre metrics began with the discovery of the generalized Runge–Lenz vector for the Taub–NUT metric [1] and the derivation of its Killing–Stäckel tensor in [2]. It was generalized to the Eguchi–Hanson metric in [3] where the Hamilton–Jacobi equation was separated. A further progress led to the integrability proof of the full 2-centre metric [2] which includes Taub–NUT and Eguchi–Hanson as particular cases. Despite these successes, a systematic analysis of the full family of the multi-centre metrics was still lacking. We will present a new approach which will determine which metrics, in this class, do exhibit a quadratic Killing–Stäckel tensor leading to classical integrability.

2 Basic facts on the multi-centre metrics

These Euclidean and four dimensional metrics on M_4 have at least one Killing vector $\widetilde{\mathcal{K}} = \partial_t$. Fibrating along this Killing vector, we have for metric

$$g = \frac{1}{V}\,(dt+\Theta)^2 + V\,d\vec{x}\cdot d\vec{x}, \qquad V = V(x), \qquad \Theta = \Theta_i(x)\,dx^i, \tag{1}$$

where the x^i are the coordinates of the subspace orthogonal to the $U(1)$ fibers. The metric (1) is a solution of the empty space Einstein equations provided that we have the monopole equation

$$dV = \star\, d\Theta, \tag{2}$$

where the Hodge star refers to the flat 3-dimensional metric. Notice that the integrability condition for the monopole equation is $\Delta V = 0$. So to each solution of Laplace equation in a 3-dimensional *flat* space there corresponds a Ricci-flat four-dimensional metric: we have an exact linearization of the empty space Einstein equations!

These metrics have a richer geometry since they are endowed with a triplet of complex structures, with associated closed 2-forms

$$\Omega_i^{(-)} = E_0\wedge E_i - \frac{1}{2}\,\epsilon_{ijk}\,E_j\wedge E_k = (dt+\Theta)\wedge dx_i - \frac{1}{2}\,V\,\epsilon_{ijk}\,dx_j\wedge dx_k. \tag{3}$$

It follows that the multi-centre metrics are hyper-Kähler, hence Ricci-flat.

As shown by [4], in the hyper-Kähler case, any Killing vector has to be either tri-holomorphic or holomorphic with respect to the triplet of complex structures (3). For instance $\widetilde{\mathcal{K}}$ is tri-holomorphic and this means that

$$\underset{\widetilde{\mathcal{K}}}{\mathcal{L}}\,\Omega_i^{(-)} = 0, \qquad i = 1, 2, 3,$$

whereas for an holomorphic Killing vector K we would have

$$\underset{K}{\mathcal{L}}\,\Omega_1^{(-)} = \Omega_2^{(-)}, \qquad \underset{K}{\mathcal{L}}\,\Omega_2^{(-)} = -\Omega_1^{(-)}, \qquad \underset{K}{\mathcal{L}}\,\Omega_3^{(-)} = 0.$$

The following drawings display the potentials for the general 2-centre metrics. For generic parameters we have two Killing vectors (the subscripts H or T refer to its holomorphic or tri-holomorphic nature). For special values of the parameters the isometry algebra is enhanced to $u(2)$ leading to either Taub–NUT or to Eguchi–Hanson metrics.

$$\left\{\begin{array}{l} \bullet \quad V = v_0 + \dfrac{m_1}{|\vec{x} - c\vec{e}_z|} + \dfrac{m_2}{|\vec{x} + c\vec{e}_z|} \\[2ex] \bullet \quad \text{isometries}: \ u(1)_{\mathrm{H}} \oplus u(1)_{\mathrm{T}} \end{array}\right.$$

$m_1 = m_2 \to m$

$c \to 0$

$m_1 = m_2 \to 1$

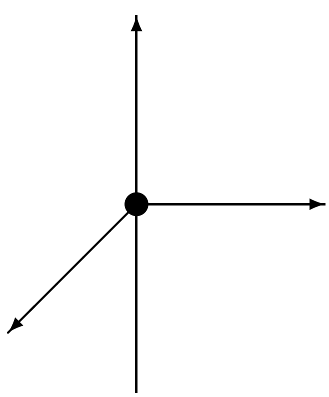
$v_0 \to 0$

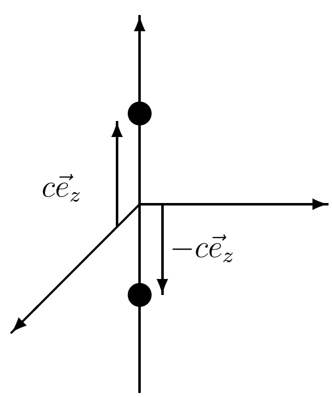
$\boxed{\text{Taub–NUT}}$

$$\left\{\begin{array}{l} \bullet \quad V = v_0 + \dfrac{m}{|\vec{x}|} \\[2ex] \bullet \quad \text{isometries}: \ su(2)_{\mathrm{H}} \oplus u(1)_{\mathrm{T}} \\[2ex] \bullet \quad a \ \text{breaks} \ su(2)_{\mathrm{T}} \\[2ex] \bullet \quad \text{flat iff} \ v_0 = 0 \ \text{or} \ m = 0 \end{array}\right.$$

$\boxed{\text{Eguchi–Hanson}}$

$$\left\{\begin{array}{l} \bullet \quad V = \dfrac{1}{|\vec{x} - c\vec{e}_z|} + \dfrac{1}{|\vec{x} + c\vec{e}_z|} \\[2ex] \bullet \quad \text{isometries}: \ u(1)_{\mathrm{H}} \oplus su(2)_{\mathrm{T}} \\[2ex] \bullet \quad c \ \text{breaks} \ su(2)_{\mathrm{H}} \\[2ex] \bullet \quad \text{flat iff} \ c = 0 \end{array}\right.$$





2.1 Geodesic flow

The geodesic flow is the Hamiltonian flow of the metric considered as a function on the cotangent bundle of M_4. Using the coordinates (t, x_i) we will write a cotangent vector as

$$\Pi_i\, dx_i + \Pi_0\, dt,$$

and the symplectic form is

$$dx_i \wedge d\Pi_i + dt \wedge d\Pi_0.$$

We take for Hamiltonian

$$H = \frac{1}{2}\, g^{\mu\nu}\, \Pi_\mu\, \Pi_\nu == \frac{1}{2}\left(\frac{1}{V}\,(\Pi_i - \Pi_0\,\Theta_i)^2 + V\,\Pi_0^2\right).$$

For geodesics orthogonal to the $U(1)$ fibers and affinely parametrized by λ the equations for the flow allow on the one hand to express the velocities

$$\begin{aligned} \dot{t} &\equiv \frac{dt}{d\lambda} = \frac{\partial H}{\partial \Pi_0} = \left(V + \frac{\Theta^2}{V}\right)\Pi_0 - \frac{\Theta_i \Pi_i}{V}, \\ \dot{x}_i &\equiv \frac{dx_i}{d\lambda} = \frac{\partial H}{\partial \Pi_i} = \frac{1}{V}\, p_i, \qquad p_i = \Pi_i - \Pi_0\,\Theta_i, \end{aligned} \tag{4}$$

and on the other hand to get the dynamical evolution equations

$$\dot{\Pi}_0 = -\frac{\partial H}{\partial t} = 0, \qquad q \equiv \Pi_0 = \frac{(\dot{t} + \Theta_i\,\dot{x}_i)}{V}, \tag{5}$$

and

$$\dot{\Pi}_i = -\frac{\partial H}{\partial x_i} \quad\Longrightarrow\quad \dot{p}_i = \left(\frac{H}{V} - q^2\right)\partial_i V + \frac{q}{V}\,(\partial_i \Theta_s - \partial_s \Theta_i)\, p_s. \tag{6}$$

Relation (5) expresses the conservation of the charge q, a consequence of the $U(1)$ isometry of the metric. For the multi-centre metrics, use of relation (2) brings the equations of motion (6) to the nice form

$$\dot{\vec{p}} = \left(\frac{H}{V} - q^2\right)\vec{\nabla}\, V + \frac{q}{V}\,\vec{p} \wedge \vec{\nabla} V. \tag{7}$$

The conservation of the energy

$$H = \frac{1}{2}\left(\frac{p_i^2}{V} + q^2\, V\right) = \frac{V}{2}\left(\dot{x}_i^2 + q^2\right) = \frac{1}{2}\, g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu$$

is obvious since it expresses the constancy of the length of the tangent vector $\dot{x}^\mu$ along a geodesic.

Hence at this stage we have already obtained two conserved quantities: $q = \Pi_0$ and H. Two more quantities are needed for integrability.

2.2 Killing–Stäckel tensors and their conserved quantities

A Killing–Stäckel (KS) tensor is a symmetric tensor $S_{\mu\nu}$ which satisfies

$$\nabla_{(\mu}\, S_{\nu\rho)} = 0. \tag{8}$$

Let us observe that if K and L are two (possibly different) Killing vectors their symmetrized tensor product $K_{(\mu}\, L_{\nu)}$ is a KS tensor. So we will define *irreducible* KS tensors as the ones which cannot be written as linear combinations, with constant coefficients, of symmetrized tensor products of Killing vectors.

For a given KS tensor $S_{\mu\nu}$ the quadratic form of the velocities:

$$\mathcal{S} = S_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu} \tag{9}$$

is preserved by the geodesic flow.

In all what follows we will look for KS tensors preserved by Lie dragging along the tri-holomorphic Killing vector:

$$\underset{\widetilde{\mathcal{K}}}{\mathcal{L}}\,S_{\mu\nu} = 0, \qquad \widetilde{\mathcal{K}} = \partial_t,$$

and we will consider *generic* values of H and $q \neq 0$.

Let us examine more closely the structure of the conserved quantity induced by such a KS tensor. Expanding the sums in (9) and using the relations (4) we obtain the following structure for the conserved quantity:

$$\mathcal{S} = A_{ij}(x_k)\,p_i\,p_j + 2q\,B_i(x_k)\,p_i + C(x_k), \tag{10}$$

where the various unknown functions, as a consequence of our hypotheses, are independent of the coordinate on the $U(1)$ fiber.

It is interesting to notice that the knowledge of $\mathcal{S}$ is *equivalent* to the knowledge of the KS tensor.
Imposing the conservation of $\mathcal{S}$ under the geodesic flow gives the following equations[1]

$$\begin{aligned}
&a) \quad q\cdot\underset{B}{\mathcal{L}}\,V = 0,\\
&b) \quad \partial_{(k}\,A_{ij)} = 0,\\
&c) \quad q(\partial_{(i}\,B_{j)} - A_{s(i}\,\epsilon_{j)su}\,\partial_u V) = 0,\\
&d) \quad \partial_i\,C + 2\left(H - q^2\,V\right)A_{is}\,\partial_s\,V - 2\,q^2\,\epsilon_{ist}\,B_s\,\partial_t V = 0.
\end{aligned} \tag{11}$$

The relation (11a) calls for a spatial Killing vector for the potential V. However it could well be that V has no symmetry at all. In this event we must have $B_i = 0$. Then relation (11c) can be written

$$[A, R] = 0, \qquad (R)_{ij} = \epsilon_{isj}\,\partial_s\,V.$$

Since V has no Killing the matrix R is a generic matrix in the Lie algebra $so(3)$. By Schur lemma it follows that A has to be proportional to the identity matrix and this does trivialize the corresponding conserved quantity $\mathcal{S}$. So the potential must have at least one extra spatial Killing vector L (notice that L lifts up to an isometry of the 4 dimensional metric).

The full discussion and integration of the system (11) has been worked out in [5] and the results can be classified according to wether the extra Killing vector is holomorphic or tri-holomorphic. For each potential the detailed form of the conserved quantity induced by the Killing–Stäckel tensor has been determined. Let us state these results.

3 One extra holomorphic Killing vector

We can take for extra spatial Killing vector $\widehat{\mathcal{L}} = x\partial_y - y\partial_x$. Then the metrics with the following potentials will exhibit a quadratic Killing–Stäckel tensor:

[1]Since H and q take generic values, the quantity $H - q^2V$ does not vanish identically.

1. The 2-centre metric with

$$V = v_0 + \frac{m_1}{r_+} + \frac{m_2}{r_-}, \qquad r_\pm = \sqrt{x^2 + y^2 + (z \pm c)^2}.$$

The conserved quantity is

$$\mathcal{S}_I = \vec{L}^{\,2} + c^2\, p_z^2 + 2\, qc\, \Delta\, L_z + 2cz\, \Delta\, \big(H - q^2 V\big) - q^2 r^2 \Delta^2, \qquad r^2 = x^2 + y^2 + z^2,$$

with

$$L_i = \epsilon_{ijk}\, x_j\, p_k, \qquad \Delta = \frac{m_1}{r_+} - \frac{m_2}{r_-}.$$

which we have checked to be in full agreement with [2].

2. A first dipolar breaking of Taub–NUT with

$$V = v_0 + \frac{m}{r} + F\frac{z}{r^3}, \qquad r = \sqrt{x^2 + y^2 + z^2},$$

which is, in fact, the singular limit $c \to 0$ of the previous case. The conserved quantity is

$$\mathcal{S}_{II} = \vec{L}^{\,2} - 2\, q\, \frac{F}{r}\, L_z - 2F\, \frac{z}{r}\, \big(H - q^2 v_0\big) + q^2\, F^2 \frac{\big(x^2 + y^2\big)}{r^4}.$$

3. A second dipolar breaking of Taub–NUT with

$$V = v_0 + \frac{m}{r} + Ez.$$

The conserved quantity is

$$\mathcal{S}_{III} = (\vec{p} \wedge \vec{L})_z - q\, \left(\frac{m}{r} - E\, z\right) L_z - 2U\, \big(H - q^2 v_0\big) - 2q^2\, mE\, \frac{\big(x^2 + y^2\big)}{r}.$$

In the limiting case of Taub–NUT, we have in fact 3 extra Killing vectors, which are holomorphic. It follows that we will have 3 Killing–Stäckel tensors, giving rise to the generalized Runge–Lenz vector

$$\vec{\mathcal{S}} = \vec{p} \wedge \vec{J} + m\, \big(q^2 v_0 - H\big)\, \frac{\vec{r}}{r}, \qquad \vec{J} = \vec{L} + q\, \frac{m}{r}\, \vec{r},$$

discovered by Gibbons and Manton [1]. The conserved quantity $\mathcal{S}_{III}$ does reduce to $\mathcal{S}_z$ in the limit $E \to 0$.

All of these metrics have at least the 4 conserved quantities

$$q = \Pi_0, \qquad H, \qquad L^\mu \Pi_\mu, \qquad \mathcal{S},$$

which are easily checked to be in involution, thus establishing the Liouville integrability of their corresponding geodesic flows.

Let us consider the class of metrics with an extra tri-holomorphic Killing.

4 One extra tri-holomorphic Killing

Here we can take the Killing vector to be $\widehat{L} = \partial_z$ and since the potential does depend only on the coordinates x and y we will state the results using the complex coordinate $z = x + iy$. Then the metrics with the following potentials will exhibit a quadratic Killing–Stäckel tensor:

1. First case:

$$V = v_0 + m\frac{z}{\sqrt{z^2+c^2}} + \text{c.c.},$$

where c.c. means complex conjugation. The conserved quantity is

$$\mathcal{S}_1 = L_z^2 - c^2\,\Pi_x^2 - 2c^2F\,\Pi_0\,\Pi_z + c^2(2v_0U + D)\Pi_0^2 - 2c^2UH, \qquad c \neq 0,$$

with

$$U + iF = -m\,\frac{z+\overline{z}}{\sqrt{z^2+c^2}}, \qquad D = -2|m|^2\,\frac{\left(z^2+\overline{z}^2+|z|^2+c^2\right)}{|\sqrt{z^2+c^2}|^2}.$$

The Bianchi VII_0 and Bianchi VI_0, which appear as particular cases of these metrics for $v_0 = 0$ were already known to be integrable from [6].

2. Second case:

$$V = v_0 + \frac{m}{2z^2} + \text{c.c.}, \qquad m \in \mathbb{C},$$

with the conserved quantity

$$\mathcal{S}_2 = L_z^2 - 2F\,\Pi_0\,\Pi_z + 2v_0U\,\Pi_0^2 - 2UH, \qquad U + iF = m\,\frac{\overline{z}}{z}.$$

3. Third case:

$$V = v_0 + \frac{m}{\sqrt{z}} + \text{c.c.}, \qquad m \in \mathbb{C},$$

with the conserved quantity

$$\mathcal{S}_3 = \Pi_y\,L_z - 2F\,\Pi_0\,\Pi_z + (2v_0U + D)\Pi_0^2 - 2UH,$$

and

$$U + iF = -\frac{m}{2}\,\frac{z-\overline{z}}{\sqrt{z}}, \qquad D = |m|^2\,\frac{z+\overline{z}}{|\sqrt{z}|^2}.$$

4. The fourth case:

$$V = v_0 + m(z+\overline{z})/2, \qquad m \in \mathbb{R},$$

is super-integrable, with two conserved quantities

$$\mathcal{S}_4^{(1)} = \Pi_y^2 + (\Pi_z - my\,\Pi_0)^2, \qquad \mathcal{S}_4^{(2)} = \Pi_x\,\Pi_y - V\,\Pi_0(\Pi_z - my\,\Pi_0) - my\,H.$$

The parameter v_0 is always real.

Here too, all of these metrics have at least the 4 conserved quantities

$$q = \Pi_0, \qquad H, \qquad L^\mu\Pi_\mu, \qquad \mathcal{S},$$

which are easily checked to be in involution, thus establishing the Liouville integrability of their corresponding geodesic flows.

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